A THEORETICAL AND EMPIRICAL ANALYSIS OF THE LIBOR MARKET MODEL AND ITS APPLICATION IN THE SOUTH AFRICAN SAFEX JIBAR MARKET

by

VICTOR GUMBO

submitted in accordance with the requirements for the

degree of

DOCTOR OF PHILOSOPHY

in the subject

OPERATIONS RESEARCH

at the

UNIVERSITY OF SOUTH AFRICA

PROMOTER: PROF B SWART

March 2007

Acknowledgements

Special thanks go to my Promoter for having dedicated quality time to the success of this work.

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Introduction

Instantaneous rate models, although theoretically satisfying, are less so in practice. Instantaneous rates are not observable and calibration to market data is complicated. Hence, the need for a market model where one models LIBOR rates seems imperative. In this modeling process, we aim at regaining the Black-76 formula[7] for pricing caps and floors since these are the ones used in the market. To regain the Black-76 formula we have to model the LIBOR rates as log-normal processes. The whole construction method means calibration by using market data for caps, floors and swaptions is straightforward. Brace, Gatarek and Musiela[8] and, Miltersen, Sandmann and Sondermann[25] showed that it is possible to construct an arbitrage-free interest rate model in which the LIBOR rates follow a log-normal process leading to Black-type pricing formulae for caps and floors. The key to their approach is to start directly with modeling observed market rates, LIBOR rates in this case, instead of instantaneous spot rates or forward rates. Thereafter, the market models, which are consistent and arbitrage-free[6], [22], [8], can be used to price more exotic instruments. This model is known as the LIBOR Market Model.

In a similar fashion, Jamshidian[22] (1998) showed how to construct an arbitrage-free interest rate model that yields Black-type pricing formulae for a certain set of swaptions. In this particular case, one starts with modeling forward swap rates as log-normal processes. This model is known as the Swap Market Model.

Some of the advantages of market models as compared to other traditional models are that market models imply pricing formulae for caplets, floorlets or swaptions that correspond to market practice. Consequently, calibration of such models is relatively simple[8].

The plan of this work is as follows. Firstly, we present an empirical analysis of the standard risk-neutral valuation approach, the forward risk-adjusted valuation approach, and elaborate the process of computing the forward risk-adjusted measure. Secondly, we present the formulation of the LIBOR and Swap market models based on a finite number of bond prices[6], [8]. The technique used will enable us to formulate and name a new model for the South African market, the SAFEX-JIBAR model.

In [5], a new approach for the estimation of the volatility of the instantaneous short interest rate was proposed. A relationship between observed LIBOR rates and certain unobserved instantaneous forward rates was established. Since data are observed discretely in time, the stochastic dynamics for these rates were determined under the corresponding risk-neutral measure and a filtering estimation algorithm for the time-discretised interest rate dynamics was proposed.

Thirdly, the SAFEX-JIBAR market model is formulated based on the assumption that the forward JIBAR rates follow a log-normal process. Formulae of the Black-type are deduced and applied to the pricing of a Rand Merchant Bank cap/floor. In addition, the corresponding formulae for the Greeks are deduced. The JIBAR is then compared to other well known models by numerical results.

Lastly, we perform some computational analysis in the following manner. We generate bond and caplet prices using Hull's [19] standard market model and calibrate the LIBOR model to the cap curve, i.e determine the *implied volatilities* σ_i's which can then be used to assess the volatility most appropriate for pricing the instrument under consideration. Having done that, we calibrate the Ho-Lee model to the bond curve obtained by our standard market model. We numerically compute caplet prices using the Black-76 formula for caplets and compare these prices to the ones obtained using the standard market model. Finally we compute and compare swaption prices obtained by our standard market model and by the LIBOR model.

Chapter 1

Probability Measures

The market price of risk of a variable determines the growth rates of all securities dependent on the variable. As we move form one market price of risk to another, the expected growth rates of security prices change, but their volatilities remain the same. Choosing a particular market price of risk is also referred to as defining the *probability measure*.

1.1 Risk Neutral Probability Measures:- Discrete Case

1.1.1 Discrete Single-period market

Consider a discrete single-period securities market with the following model specifications[28].

- Initial date $t = 0$, terminal date $t = 1$.

- A finite sample space $\Omega = \{\omega_1, \omega_2, \ldots, \omega_k\}$ where $\omega_i,\ i = 1, \ldots, k$ is a possible state of the world realized at time $t = 1$.

- A probability measure P on Ω.

- A bank account process $B_t(\omega_i),\ t = 0, 1$ with $B_0 = 1$ and $B_1 \geq 1$.

- Define interest rate as $r = \frac{B_1 - B_0}{B_0} = B_1 - 1$ which is deterministic if B_1 is deterministic.

- A price process $S_n(t) = \{S_n(0), S_n(1)\}$ for risky securities S_n, $n = 1, \ldots, N$.

- A trading strategy $H = (H_0, H_1, \ldots, H_N)$ determining an investor's portfolio over the period.

- A value process $V_t = \{V_0, V_1\}$ giving the total value of the portfolio. It is given by

$$V_t = H_0 B_t + \sum_{n=1}^{N} H_n S_n(t). \tag{1.1}$$

- A gains process G giving the profit/loss generated by the portfolio over the period, where

$$G = V_1 - V_0 = H_0 r + \sum_{n=1}^{N} H_n \Delta S_n, \tag{1.2}$$

with $\Delta S_n = S_n(1) - S_n(0)$.

To compare the movement in prices, we need a reference asset. We get normalized price processes by discounting prices at time $t = 1$ with respect to the bank account B_1 . This normalization process is called discounting and the bank account process is then called the numeraire. We then define the following:

- A discounted price process S^* where $S^*(0) = S(0)$ and $S^*(1) = (\frac{S_1(1)}{B_1}, \ldots, \frac{S_N(1)}{B_1})$.

- A discounted value process $V_t^* = V_t/B_t$.

- A discounted gains process $G^* = V^*{}_1 - V^*{}_0 = \sum_{n=1}^{N} H_n \Delta S_n{}^*$.

Definition 1.1 *An arbitrage opportunity is a trading strategy* H *such that:*

(i) $V_0 = 0,$

(ii) $V_1 \geq 0,$

(iii) $E_P[V_1] > 0.$

Definition 1.2 *A non-negative linear probability measure* Q *on* Ω

is a risk-neutral probability measure iff

(i) $Q(w) > 0, \forall \omega \in \Omega,$ *and*

(ii) $E_Q[\Delta S^*{}_n] = 0, n = 1, 2, \ldots, N,$ *i.e* $E_Q[S^*{}_n(1)] = S^*{}_n(0)$.

We state without proof the following theorem[28].

Theorem 1.3 *There are no arbitrage opportunities if and only if there exists a risk-neutral probability measure Q.*

Proposition 1.4 *If Q is any risk-neutral probability measure, then for every trading strategy H one has $V_0 = E_Q[V_1/B_1]$.*

Proof

$$
\begin{aligned}
V_0 = V^*{}_0 = E_Q[V^*{}_0] &= E_Q[V^*{}_1 - G^*] \\
&= E_Q[V^*{}_1] - E_Q[\sum_{n=1}^{N} H_n \Delta S_n{}^*] \\
&= E_Q[V^*{}_1] - \underbrace{\sum_{n=1}^{N} H_n E_Q[\Delta S_n{}^*]}_{=0} \\
&= E_Q[V^*{}_1] \\
&= E_Q[V_1/B_1].
\end{aligned}
$$

The statement in the underbrace gives zero because of the fact that Q is a risk-neutral measure and (ii) of Definition 1.2 applies. The subscript Q in E_Q indicates that we are using the specifically computed no-arbitrage pricing probability Q, commonly known as the risk-neutral probability measure.

A contingent claim is a random variable X representing a payoff at terminal time, say $t = 1$. It is said to be attainable or marketable if there is some trading strategy H, called the replicating portfolio, such that $V_1 = X$.

Attainability of a contingent claim means that its initial value remains constant for every risk-neutral probability measure. In fact, we have the following theorem.

Theorem 1.5 (Risk Neutral Valuation Principle) *If the single-period market is free of arbitrage opportunities, then the time $t = 0$ value of an attainable contingent claim X is $E_Q[X/B_1]$, where Q is any risk-neutral probability measure.*

Clearly, the above theorem says that by the attainability of X we mean that there exists a portfolio V with $V_1 = X$ such that

$$\begin{aligned} V_0 &= E_Q[V_1/B_1] \\ &= E_Q[X/B_1] \\ &= X_0 \end{aligned} \tag{1.3}$$

since $V_0 = X_0$ by the no-arbitrage argument.

The measure P in Definition 1.1, though it defines arbitrage, plays no role in the pricing of claims.

As an example to illustrate what we have seen so far, consider the following problem:

[28]Suppose $K = 2, N = 1$ and the interest rate is a scalar parameter $r \geq 0$. Also, suppose $S_0 = 1, S_1(\omega_1) = u$ and $S_1(\omega_2) = d$, $u > d > 0$. For what values of r, u and d does there exist a risk neutral probability measure? Otherwise, what arbitrage opportunities are there?

Solution

$S_1(0) = 1, S_1(1, \omega_1) = u, S_1(1, \omega_2) = d$ and therefore ${S_1}^*(1, \omega_1) = u/(r+1), {S_1}^*(1, \omega_2) = d/(r+1)$. Let $Q(\omega)$ be a risk neutral probability measure. Thus from the definition of a probability and the risk neutral valuation principle, we need to solve the system:

$$\begin{aligned} Q(\omega_1) + Q(\omega_2) &= 1 \\ \frac{u}{r+1}Q(\omega_1) + \frac{d}{r+1}Q(\omega_2) &= 1 \end{aligned}$$

from which we get the solution:

$$\begin{aligned} Q(\omega_1) &= \frac{r+1-d}{u-d} \\ Q(\omega_2) &= \frac{u-r-1}{u-d}. \end{aligned}$$

There exists a risk neutral probability measure $Q = (\frac{r+1-d}{u-d}; \frac{u-r-1}{u-d})$ if $Q > 0$, that is if $d < r+1 < u$ with $u - d > 0$.

For any other values not satisfying the above, there are arbitrage opportunities which we calculate below.

Let $H = (H_0, H_1)$ be the trading strategy. Then we wish to find (H_0, H_1) satisfying $V_0 = 0$ and $V_1 \geq 0$, with at least one $V_1(\omega_i) > 0$.

$$\begin{aligned} V_0 &= H_0 + H_1 S_1(0) \\ &= H_0 + H_1 = 0 \Rightarrow H_0 = -H_1. \end{aligned}$$

But the discounted value process is given by:

$$V_1{}^* = H_0 + H_1 S_1{}^*(1).$$

Thus

$$\begin{aligned} V_1{}^*(\omega_1) &= H_0 + \frac{u}{r+1} H_1 \Rightarrow V_1{}^*(\omega_1) = H_1(\frac{u}{r+1} - 1) \\ V_1{}^*(\omega_2) &= H_0 + \frac{d}{r+1} H_1 \Rightarrow V_1{}^*(\omega_2) = H_1(\frac{d}{r+1} - 1). \end{aligned}$$

So the arbitrage opportunities are all the situations $H = (H_0, H_1)$ satisfying

$H_0 = -H_1$; $H_1(\frac{u}{r+1} - 1) \geq 0$; $H_1(\frac{d}{r+1} - 1) \geq 0$;, with at least one equality being strict.

A market in which every contingent claim X can be generated by some trading strategy is called a complete market. Otherwise it is called an incomplete market. In a complete market, all the fundamental goods or instruments have a fair price.

1.1.2 The multiperiod case

With all the definitions from the above section, consider $t = 0, 1, \ldots, T$.

- A finite sample space $\Omega = \{\omega_1, \ldots, \omega_k\}$.

- A filtration $\mathbf{F} = \{\mathcal{F}_t; t = 0, \ldots, T\}$.

- N risky security processes $S_n = \{S_n(t); t = 0, \ldots, T\}$, where $S_n(t)$ is a non-negative stochastic process for each $n = 1, 2, \ldots, N$, adapted to $\mathbf{F}$.

- A bank process B_t.

- $\mathcal{P}_t$, a time-t partition of Ω.

In many cases, money cannot be added to or withdrawn from a portfolio except at time $t = 0$ and time $t = T$. Such is the case of a fixed account. In such cases, any change in the portfolio value is attributed to internal gains and losses in the instrument. Such a portfolio receives the name of "self-financing" portfolio. Mathematically, the value of such a portfolio at any given time t is given by

$$V_t = H_0(t+1)B_t + \sum_{n=1}^{N} H_n(t+1)S_n(t), t = 1, \ldots, T-1,$$

where H is the trading strategy.

In the multi-period case, a trading strategy $H = (H_0, H_1, \ldots, H_N)$ is an $N+1$-dimensional vector whose components are stochastic processes of the form

$$H_n = \{H_n(t); t = 1, 2, \ldots, T\}, n = 0, 1, \ldots, N.$$

Each H_n is said to be predictable with respect to the filtration $\mathbf{F}$ if each $H_n(t)$ is measurable with respect to $\mathcal{F}_{t-1}, \forall t = 1, 2, \ldots, T$. Since $\mathcal{F}_{t-1} \subseteq \mathcal{F}_t$, all predictable stochastic processes are adapted. An adapted stochastic process

$$H_n = \{H_n(t); t = 1, 2, \ldots, T\}$$

is called a supermartingale if

$$E_Q[H_n(t+s)|\mathcal{F}_t] \leq H_n(t), \ \forall s, t \geq 0,$$

and is called a submartingale if

$$E_Q[H_n(t+s)|\mathcal{F}_t] \geq H_n(t),\ \forall s,t \geq 0.$$

If equality holds, then H_n is called a martingale and Q is then called a martingale measure.

Theorem 1.6 (Risk neutral valuation principle) *[28] The time t value of a marketable contingent claim X is equal to V_t, the time t value of the portfolio which replicates X. Moreover,*

$$V_t{}^* = V_t/B_t = E_Q[X/B_t|\mathcal{F}_t], t = 0, 1, \ldots, T \tag{1.4}$$

for all risk-neutral probability measures Q.

In view of what we know for single-period models, corresponding to each underlying single period model is a risk-neutral probability measure. For example, corresponding to each $A \in \mathcal{P}_t$ for $t < T$ there is a probability measure, denoted by $Q(t, A)$, on the single period space. This probability measure gives positive mass to each cell $A' \subseteq A$ in the partition $\mathcal{P}_{t+1}$, sums to one over such cells, and satisfies $E_{Q(t,A)}[\Delta S_n{}^*(t+1)] = 0$, for each $n = 1, \ldots, N$.
$Q(t, A)$ is a conditional risk-neutral probability such that

$$E_Q[S^*(t+1)|\mathcal{P}_t] = E_{Q(t,A)}[S^*(t+1)] = S^*(t).$$

This equation defines Q in terms of $Q(t, A)$.

Theorem 1.7 *The probability measure Q is a martingale.*

Proof

Since $E_{Q(t,A)}[\Delta S_n{}^*(t+1)] = 0$ for $n = 1, \ldots, N; A \in \mathcal{P}_t$ and $t < T$, it follows from the construction of Q that

$$E_Q[\Delta S_n{}^*(t+1)|\mathcal{F}_t] = 0, \ n = 1, \ldots, N, \ t < T. \tag{1.5}$$

Taking arbitrary $s, t \geq 0$ and n and using the above equation:

$$\begin{aligned}
E_Q[S_n{}^*(t+s)|\mathcal{F}_t] &= E_Q[\Delta S_n{}^*(t+s) + \ldots + \Delta S_n{}^*(t+1) + S_n{}^*(t)|\mathcal{F}_t] \\
&= E_Q[E_Q[\Delta S_n{}^*(t+s)|\mathcal{F}_{t+s-1}]|\mathcal{F}_t] + \ldots \\
&+ E_Q[E_Q[\Delta S_n{}^*(t+1)|\mathcal{F}_t]|\mathcal{F}_t] + S_n{}^*(t) \\
&= E_Q[0|\mathcal{F}_t] + \ldots + E_Q[0|\mathcal{F}_t] + S_n{}^*(t) \\
&= S_n{}^*(t).
\end{aligned}$$

Hence Q is a martingale measure.

We show through an example[28] how to compute Q in a multi-period environment with more than two states of the world.

Consider a simple model with $T = 2$ and $K = 4$. Suppose $r = 0$ and the risky security is as follows:

ω_k	$t=0$	$t=1$	$t=2$
ω_1	$S_0=5$	$S_1=8$	$S_2=9$
ω_2	$S_0=5$	$S_1=8$	$S_2=6$
ω_3	$S_0=5$	$S_1=4$	$S_2=6$
ω_4	$S_0=5$	$S_1=4$	$S_2=3$

Let (Q_u, Q_d) be the martingale measure in the first period. Since $r=0$, the discounted price process is equal to the price process. To find (Q_u, Q_d) we solve the linear system:

$$Q_u + Q_d = 1$$

$$8Q_u + 4Q_d = 5$$

which gives $(Q_u, Q_d) = (1/4, 3/4)$. In period 2 we solve two systems namely,

$$Q_{uu} + Q_{ud} = 1$$

$$9Q_{uu} + 6Q_{ud} = 8$$

and

$$Q_{du} + Q_{dd} = 1$$

$$6Q_{du} + 3Q_{dd} = 4.$$

The solutions to these systems are

$$\begin{aligned} (Q_{uu}, Q_{ud}) &= (2/3, 1/3) \\ (Q_{du}, Q_{dd}) &= (1/3, 2/3). \end{aligned}$$

Then the risk neutral probability measure (or martingale measure) is

$$\begin{aligned} Q &= (Q_u Q_{uu}, Q_u Q_{ud}, Q_d Q_{du}, Q_d Q_{dd}) \\ &= (1/6, 1/12, 1/4, 1/2). \end{aligned}$$

1.2 Forward Risk Neutral Probability Measures - Discrete Case

We follow the exposition in [28].

1.2.1 A fundamental probability relation

For some $\tau \leq T$, consider a random variable $M_\tau \in \mathcal{F}_\tau,\ M_\tau > 0$ such that $E_Q[M_\tau] = 1$ for some risk neutral probability measure Q. Now, define

$$P_\tau \equiv M_\tau(\omega) Q(\omega), \forall \omega \in \Omega.$$

Clearly P_τ is a legitimate measure :

$E_Q[M_\tau] = 1,\ Q(\omega) > 0 \Rightarrow P_\tau(\Omega) = 1,\ P_\tau(\omega) > 0.$

Let E_τ be the expectation operator corresponding to P_τ. Define

$M = \{M_t : t = 0, \ldots, \tau\}$ with $M_t = E_Q[M_\tau|\mathcal{F}_t], t = 0, \ldots, \tau.$

That is, by construction $\{M_t\}$ is a martingale w.r.t Q, and $M_t \in \mathcal{F}_t$.

Also, $M_0 = E_Q[M_\tau|\mathcal{F}_0] = E_Q[M_\tau] = 1.$

Theorem 1.8 *If X is a random variable, then $E_\tau[M_tX|\mathcal{F}_t] = E_Q[M_\tau X|\mathcal{F}_t],\ t = 0, \ldots, \tau.$*

Proof

- Suppose $X \in \mathcal{F}_t$. Then $M_tX = XE_Q[M_\tau|\mathcal{F}_t]$, and the theorem is proved. Otherwise:

- For $t = 0, M_0 = 1$ and

$$\begin{aligned}
E_\tau[M_0X|\mathcal{F}_0] &= E_\tau[1 \cdot X|\mathcal{F}_0] \\
&= E_\tau[X] \\
&= \sum_\omega X(\omega)P_\tau(\omega) \\
&= \sum_\omega X(\omega)M_\tau(\omega)Q(\omega),\ \forall\omega \in \Omega \\
&= E_Q[M_\tau X] \\
&= E_Q[M_\tau X|\mathcal{F}_0].
\end{aligned}$$

• For the general case, take $A \in \mathcal{P}_t$, and then show that

$E_\tau[M_t X|A] = E_Q[M_\tau X|A]$.

In this case,

$$\begin{aligned} E_\tau[M_t X|A] &= \frac{\sum_{\omega \in A} X(\omega) M_t(\omega) P_\tau(\omega)}{\sum_{\omega \in A} P_\tau(\omega)} \\ &= \frac{\sum_{\omega \in A} X(\omega) M_t(\omega) M_\tau(\omega) Q(\omega)}{\sum_{\omega \in A} M_\tau(\omega) Q(\omega)}. \end{aligned}$$

But, on A, M_t is constant and this constant is

$$\begin{aligned} M_t(\omega) &= E_Q[M_\tau|A], \\ &= \frac{\sum_{\omega' \in A} M_\tau(\omega') Q(\omega')}{Q(A)}, \forall \omega \in A, \end{aligned}$$

since M is a martingale with respect to Q.

Substituting for $M_t(\omega)$ and simplifying we have

$$\begin{aligned} E_\tau[M_t X|A] &= \frac{\sum_{\omega' \in A} M_\tau(\omega') Q(\omega')}{\sum_{\omega \in A} M_\tau(\omega) Q(\omega)} \frac{\sum_{\omega \in A} M_\tau(\omega) Q(\omega) X(\omega)}{Q(A)} \\ &= \frac{\sum_{\omega \in A} X(\omega) M_\tau(\omega) Q(\omega)}{Q(A)} \\ &= E_Q[M_\tau X|A]. \diamond \end{aligned}$$

Theorem 1.9 (Fundamental Relationship in probability theory)

The stochastic process $YM = \{Y_t M_t, t = 0, 1, \ldots, \tau\}$ *is a martingale under* Q *if and only if the stochastic process* $Y = \{Y_t, t = 0, \ldots, \tau\}$ *is a martingale under* P_τ.

Proof

YM is a martingale under Q if and only if $Y_tM_t = E_Q[Y_\tau M_\tau|\mathcal{F}_t],\ \forall t$. In $E_\tau[M_tX|\mathcal{F}_t] = E_Q[M_\tau X|\mathcal{F}_t]$, (Theorem 1.8) take $X = Y_\tau$ and see that

$$E_\tau[M_tY_\tau|\mathcal{F}_t] = E_Q[M_\tau Y_\tau|\mathcal{F}_t], \forall t,\ M_t \in \mathcal{F}_t.$$

But $E_\tau[M_tY_\tau|\mathcal{F}_t] = M_tE_\tau[Y_\tau|\mathcal{F}_t]$ so that YM is a martingale under Q iff

$$\begin{aligned}
E_Q[M_\tau Y_\tau|\mathcal{F}_t] &= Y_tM_t \\
\Leftrightarrow E_\tau[M_tY_\tau|\mathcal{F}_t] &= E_\tau[M_tY_\tau|\mathcal{F}_t] \\
\Leftrightarrow M_tE_\tau[Y_\tau|\mathcal{F}_t] &= Y_tM_t \\
\Leftrightarrow Y_t &= E_\tau[Y_\tau|\mathcal{F}_t], \forall t. \\
\Leftrightarrow Y & \quad \text{is a martingale under} P_\tau.
\end{aligned}$$

1.2.2 Term-structure model

We now include zero-coupon bonds in our market model.

Let $p(t,\tau)$ define the time-t price of a zero-coupon bond with maturity $\tau,\ \tau = 1,\ldots,T$ and $0 \le t \le \tau$. If we assume the model to be arbitrage-free, there must exist a risk-neutral measure Q such that

$$p(s,\tau) = E_Q[B_sp(t,\tau)/B_t|\mathcal{F}_s],\ 0 \le s \le t \le \tau.$$

or

$$p(s, \tau) = B_s E_Q[p(t, \tau)/B_t|\mathcal{F}_s].$$

That is, under a risk neutral probability measure Q, the discounted prices of the zero-coupon bonds are martingales. But $p(\tau, \tau) = 1$ and

$$B_t/B_s = (1 + r_{s+1}) \ldots (1 + r_t),$$

where r_t is the spot interest rate over the period $(t - 1, t]$. Taking $t = \tau$ we see that zero-coupon bonds must satisfy the important relationship

$$\begin{aligned} p(s, \tau) &= E_Q[B_s/B_\tau|\mathcal{F}_s] \\ &= E_Q[1/\left\{(1 + r_{s+1}) \ldots (1 + r_\tau)\right\}|\mathcal{F}_s], \end{aligned}$$

for any Q. In particular, $p(s, s+1) = \frac{1}{1+r_{s+1}}$, since r_s is a predictable process.

1.2.3 Forward prices

Suppose at time s one enters into an agreement to purchase a unit of some security at some future time t, at some future price O_s. The price O_s is called the forward price. Now suppose it is time s

and consider the forward price O_s of the asset Z (e.g a τ-maturity zero-coupon bond), delivered at time t ,where $s \leq t \leq \tau$.

Theorem 1.10 *The time s forward price O_s of security Z, which is received and paid for at time $t > s$ and which pays no dividend, is*

$$O_s = \frac{Z_s}{E_Q[B_s/B_t|\mathcal{F}_s]}. \qquad (1.6)$$

Proof

The time s cost of replicating O_s is simply the present value of O_s, that is

$$E_Q[O_s B_s/B_t|\mathcal{F}_s] = O_s E_Q[B_s/B_t|\mathcal{F}_s].$$

This is the cost of an agreement which pays out O_s at time t. You then buy and receive security Z with value Z_t at time t. Its time s present value is simply Z_s, by the definition of the martingale measure Q. So by the law of one price, the time s value of the two replicating strategies must be equal, that is, $Z_s = O_s E_Q[B_s/B_t|\mathcal{F}_s]$.$\diamond$

In view of the relations

$$p(s,\tau) \quad = \quad E_Q[B_s/B_\tau|\mathcal{F}_s], \text{ and}$$

$$O_s = \frac{Z_s}{E_Q[B_s/B_t|\mathcal{F}_s]},$$

we see that if Z is a τ-maturity zero-coupon bond, received and paid for at time t, then its forward price is

$$O_s = \frac{Z_s}{p(s,t)} \tag{1.7}$$

$$= \frac{p(s,\tau)}{p(s,t)}, \ 0 \leq s \leq t \leq \tau \leq T. \tag{1.8}$$

For the special case $\tau = t+1$, we see that

$$O_s = \frac{p(s,t+1)}{p(s,t)}, \ 0 \leq s \leq t \leq T \tag{1.9}$$

must be the time-s forward price of a zero-coupon bond that is delivered at time t and matures one period later. The yield at time s, for the bond delivered at t and maturity at $t+1$ is

$$\begin{aligned} f(s;t,t+1) &= \frac{1-O_s}{O_s} \\ &= \frac{1}{O_s} - 1 \\ &= \frac{p(s,t)}{p(s,t+1)} - 1. \end{aligned} \tag{1.10}$$

This is the LIBOR forward rate for the period $[t, t+1]$ contracted at time s and denoted simply by $f(s,t)$. Since $f(s,t)$ is associated with a single future period, it will be called the "forward spot interest rate", or simply, "forward interest rate". Clearly for $t = s$ we see that

$$f(s,s) = r_{s+1}, \ 0 \leq s \leq T$$

which is the spot rate over the time interval $(s, s+1]$.

1.2.4 Constructing the forward measures

We now return to to Section 1.2.1.

Let the stochastic process $\pi = \{\pi_t, 0 \leq t \leq s\}$ represent the price of an asset such as a stock, a zero-coupon bond or a contingent claim, where $\tau \leq s \leq T$. Set $Y_t = \pi_t/p(t,\tau)$ and recall that Y_t represents the time-t forward price for a delivery of the asset at time τ. Using our standard notation for forward prices, we therefore will sometimes write O_t for $Y_t = \pi_t/p(t,\tau)$. With Q the risk neutral probability measure, set $M_\tau = [B_\tau p(0,\tau)]^{-1}$. Note that $M_\tau(\omega) > 0$ and $E_Q[M_\tau] = [1/p(0,\tau)]E_Q[1/B_\tau] = 1$, because $p(0,\tau) = E_Q[1/B_\tau]$. Hence we can define the Q-martingale M as

$$\begin{aligned} M_t &= E_Q[M_\tau|\mathcal{F}_t] \\ &= \frac{1}{p(0,\tau)} E_Q[1/B_\tau|\mathcal{F}_t] \\ &= \frac{p(t,\tau)}{p(0,\tau)B_t} \\ &= \frac{B_0}{p(0,\tau)} \cdot \frac{p(t,\tau)}{B_t}. \end{aligned} \tag{1.11}$$

Definition 1.11 (forward risk adjusted probability measure) *We define the forward risk adjusted probability measure, also called the τ forward probability measure as*

$$\begin{aligned} P_\tau(\omega) &= M_\tau(\omega)Q(\omega) \\ &= \frac{Q(\omega)}{p(0,\tau)B_\tau(\omega)}. \end{aligned} \tag{1.12}$$

By observing that

$$\begin{aligned} Y_tM_t &\equiv O_tM_t \\ &= (\pi_t/p(t,\tau))\,p(t,\tau)/[p(0,\tau)B_t] \\ &= \frac{\pi_t}{p(0,\tau)B_t} \\ &= \frac{\pi_t}{B_t}\cdot\frac{1}{p(0,\tau)}. \end{aligned} \tag{1.13}$$

the process $(YM)_t$ represents the B_t-discounted price of the asset divided by the constant $p(0,\tau)$, and is thus a martingale under the risk neutral probability measure Q. From Theorem 1.9 we thus have that Y_t (or O_t) is a martingale under P_τ. To summarise:

Theorem 1.12 *The time-t forward price O_t for delivery of an asset at time τ is a martingale under the forward risk-adjusted probability measure P_τ, that is,*

$$O_t = E_\tau[O_\tau|\mathcal{F}_t]. \tag{1.14}$$

Now, because $O_\tau = \frac{\pi_\tau}{p(\tau,\tau)}$ we have:

$$O_t = \frac{\pi_t}{p(t,\tau)} = E_\tau[O_\tau|\mathcal{F}_t] = E_\tau[\pi_\tau|\mathcal{F}_t], t \leq \tau. \tag{1.15}$$

Multiplying through by $p(t,\tau)$ yields the following result :

Theorem 1.13 *If π_t is the time-t price of a security, then*

$$\pi_t = p(t,\tau)E_\tau[\pi_\tau|\mathcal{F}_t], \ t \leq \tau. \tag{1.16}$$

Remark: This shows that a price process π_t, *discounted* with respect to numeraire process $p(t,\tau)$, is a martingale with respect to the associated measure P_τ. To calculate the time-t price of a security , formula (1.16) only requires the conditional distribution of π_τ under the forward risk adjusted probability measure corresponding to time τ. For this reason, this new formula is applicable even in the case of stochastic interest rates, which is the case with many interest rate derivatives. This is not the case with the traditional risk neutral valuation formula which is a convenient formula when the interest rates are a known or deterministic quantity. The traditional risk-neutral valuation approach states that the time t value of a marketable contingent claim X is equal to V_t, the time t value of the portfolio which replicates X, i.e

$$V_t \;\; = \;\; B_t {V^*}_t$$

$$= \quad B_t E_Q[X/B_T|\mathcal{F}_t], t = 0, 1, \dots, T,$$

for all risk-neutral probability measures Q.

Note that the above equation is just

$$\begin{aligned} \pi_t &= B_t E_Q[\pi_\tau / B_\tau | \mathcal{F}_t] \\ &= (B_t / B_\tau) E_Q[\pi_\tau | \mathcal{F}_t]. \end{aligned}$$

1.2.5 Summary

We take time out to reiterate the steps involved in the computation of the forward risk-adjusted probability measures. These steps could be developed into an algorithm that can be illustrated numerically.

- Calculate the bond price processes using the formulae:

$$\begin{aligned} p(t, t+1) &= \frac{1}{r(t)+1}, \; t = 0, 1, \dots, T-1. \\ p(t, \tau) &= \frac{1}{r(t+1)+1} E_Q\left[p(t+1, \tau) | \mathcal{F}_t\right]. \end{aligned}$$

- Calculate the yield processes

$$Y(t, \tau) = [p(t, \tau)]^{\frac{-1}{\tau - t}} - 1, \; t = 0, 1, \dots, T-1.$$

where $Y(t, t+1) = r(t+1)$ is the current spot interest rate.

- Calculate the term structure of forward interest rates

$$f(s, t) = \frac{p(s, t)}{p(s, t+1)} - 1, \; s = 0, \dots, T-1; \; t = 0, \dots, T-1,$$

where $f(s, s) = r(s+1),\ 0 \leq s < T$, i.e forward and spot rates coincide if delivery occurs right away.

• Choose Q such that $Q(\omega) > 0,\ \forall \omega \in \Omega$.

• Compute the bank process:

$$B(t) = r(t)B(t-1) + B(t-1),\ B(0) = 1.$$

• Compute the random variable $M(t)$

$$M(t) = \frac{p(t, \tau)}{p(0, \tau)B(t)}.$$

• Finally compute the required forward risk-adjusted probability measure

$$\begin{aligned} P^{\tau}(\omega) &= M(\tau; \omega)Q(\omega) \\ &= M(t)Q(\omega). \end{aligned}$$

Chapter 2

LIBOR Market Models

2.1 Defining the LIBOR rate

Suppose we have a zero-coupon bond maturing at time T when it pays \$1. At time t it has value $p(t, T)$. Applying a constant rate of return y (yield) between t and T, one dollar received at time T has a present value of $p(t, T)$ at time t, where

$$p(t, T) = 1 \cdot e^{-y(T-t)}. \tag{2.1}$$

It follows then that the continuously compounded spot rate is given by

$$y = \frac{-\ln p(t, T)}{T - t}. \tag{2.2}$$

Consider the following situation for a T-bond. T is the maturity date, t is the contract date to invest \$1 and S is the date of investment of \$1. Let $p(t, T)$ be the price at time t of a zero-coupon bond with maturity T.

At time t, we raise $\$p(t, S)$ from the sale of one S-bond. With this income, we purchase exactly $\frac{p(t,S)}{p(t,T)}$ T-bonds which brings us to a net investment of

$$-\$p(t, S) + \frac{p(t, S)}{p(t, T)} \cdot \$p(t, T) = 0$$

at time t. When the S-bond matures, we invest \$1 and when the T-bonds mature at \$1 each, we will receive an income of $\frac{p(t,S)}{p(t,T)}$. Thus in summary, we went into an agreement at time t guaranteeing a risk-less rate of interest for the future time period $[S, T]$. Such an interest rate is called a *forward interest rate*.

Definition 2.1 (LIBOR rate) *The simple forward rate or LIBOR forward rate L for $[S, T]$ contracted at time t, is the solution to the equation*

$$1 \cdot (1 + (T - S) \cdot L) = 1 \cdot \frac{p(t, S)}{p(t, T)} \tag{2.3}$$

where time T is the maturity time of the forward LIBOR, $T - S$ is called the tenor and $1/(T - S)$ is the "accrued factor" or the "day-count fraction".

Note that

$$
\begin{aligned}
L(t, S, T) &= \frac{1}{T-S}\left[\frac{p(t, S)}{p(t, T)} - 1\right] \\
&= \frac{1}{T-S}\left[\frac{p(t, S) - p(t, T)}{p(t, T)}\right]. \qquad (2.4)
\end{aligned}
$$

If $t = S$,

$$
\begin{aligned}
L(S, T) &= L(S, S, T) \\
&= \frac{1}{T-S}\left[\frac{p(S, S) - p(t, T)}{p(t, T)}\right] \\
&= \frac{1}{T-S}\left[\frac{1 - p(t, T)}{p(t, T)}\right] \qquad (2.5) \\
&= -\frac{p(S, T) - 1}{(T-S)p(S, T)}
\end{aligned}
$$

is called the simple spot LIBOR at time S.

Similarly, with continuous compounding, the equivalent of Equation (2.3) is

$$
e^{R(T-S)} = \frac{p(t, S)}{p(t, T)}.
$$

This defines $R(t; S, T)$ and $R(S, T)$ as continuous compounded forward and spot rates respectively. Taking the limit as $S \to T$ gives $1 = \frac{p(t,T)}{p(t,T)}$. Based on certain economic assumptions, equilibrium models derive a process for the short rate and explore its effect on bond and option prices. The short rate at time t is the rate that applies to an infinitesimally short period of time at time t. It is also known as instantaneous short rate. Derivative, bond and option prices depend only on the process followed by the instantaneous short rate in a risk-neutral world.

Similarly, instantaneous forward rate is a forward rate for a very short period of time in the future.

2.1.1 LIBOR market model

Instantaneous short and forward rates are nice to handle from a theoretical point of view but have the disadvantages that they can never be observed in the market and worse still, the numerical calibration of the related models is generally complicated. Our aim is therefore not to model instantaneous rates but discrete market rates like the LIBOR rates and discrete forward swap rates to produce, respectively, formulae for caps and floors (the LIBOR models), and formulae for swaptions (the swap market models). The advantage is that these models give valuation formulae of the Black-76[7] type and hence are easily acceptable in the financial industry. Secondly, they are easy to calibrate to market data. [LIBOR stands for London Interbank Offer Rate].

In theory, the rate at which money is borrowed or lent when there is no credit risk is the risk-free rate. This rate is often thought of as the Treasury rate, i.e, the rate at which a particular government borrows in its own currency. In practice though, large financial institutions usually set this risk-free rate equal to the LIBOR. This practice is based on the fact that financial institutions invest surplus funds in the LIBOR market and borrow from this market to meet their short-term funding requirements. They regard LIBOR as their opportunity cost of capital.

The exposition in the next section is based on [6].

Caps: Definition and Market Practice

Definition 2.2 (Caplet) *A caplet is a European call option on the spot interest rate, in this case on the spot LIBOR rate L at a fixed point in time.*

Definition 2.3 (Cap) *A cap is a strip or portfolio of caplets all having common strike rate R which is the cap rate, and with one caplet for each time period in a given interval of time.*

Basically a cap is an option that protects a borrower against an increase in interest rates by giving the buyer of the cap the right but not the obligation, to borrow at an agreed rate, called the strike or cap rate, for a certain future period. If a borrower has floating interest rate liabilities and wishes to protect against an increase in short term interest rates, buying a cap would allow the borrower to limit his maximum rate of borrowing. If, however, the market rate is lower, the option is not exercised and the borrower pays the lower market rate. Thus the borrower is protected against rising interest rates but can still benefit from falling interest rates. In the South African market, settlement on the cap takes place against the 3-month JIBAR rate.

Consider a fixed set of increasing maturities $T_0,\ T_1,\ \ldots,\ T_N$ and define α_i by $\alpha_i = T_i - T_{i-1},\ i = 1, \ldots, N$. α_i is the tenor and $1/\alpha_i$ is the day-count factor. Let $p_i(t) \equiv p(t, T_i)$ and $L_i(t) \equiv L(t, T_i) \equiv L(t, T_{i-1}, T_i)$ denote the LIBOR forward rate contracted at t for the period $[T_{i-1}, T_i]$, i.e, from Equation (2.4):

$$L_i(t) = \frac{1}{T_i - T_{i-1}} \left[\frac{p_{i-1}(t) - p_i(t)}{p_i(t)} \right],\ i = 1, ..., N.$$

Definition 2.4 (Cap) *A cap with cap rate R and resettlement dates $T_0,\ T_1,\ ....,\ T_N$ is a contract which at time T_i gives the holder of the* *cap the amount*

$$X_i = \alpha_i \cdot \max\left[L_i(T_{i-1}) - R, 0\right] \tag{2.6}$$

where $L_i(t)$ *is the floating rate and* R *is the fixed strike or cap rate.* $L_i(T_{i-1}) \equiv L(T_{i-1}, T_i)$ *is in fact the spot rate for* $[T_{i-1}, T_i]$.

Thus for $i = 1, ..., N$ caplets we have the following payoffs at times $T_i,\ i = 1, \ldots, N$ respectively,

$$\begin{array}{rcl} X_1 & = & \alpha_1 \max\left[L_1(T_0) - R, 0\right] \\ X_2 & = & \alpha_2 \max\left[L_2(T_1) - R, 0\right] \\ \vdots & \ddots & \vdots \\ X_N & = & \alpha_N \max\left[L_N(T_{N-1}) - R, 0\right]. \end{array}$$

Note that the amounts X_i are already determined at times T_{i-1} but are only paid out at times T_i. The portfolio $\{X_1, X_2, ..., X_N\}$ is the cap.

Since X_i is the payoff from a call option on the underlying spot rate $L_i(T_{i-1})$, the market practice is to use the Black-76[7] formula for a call option to price a caplet.

Definition 2.5 (Black-76 formula for caplets) *The Black-76 for-*

mula (Eqn 16 of [7])for the caplet whose payoff is

$$X_i = \alpha_i \cdot \max\left[L(T_{i-1}, T_i) - R, 0\right] \tag{2.7}$$

is given by the expression

$$Capl_i^B(t) = \alpha_i p_i(t)\{L_i(t)N[d_1] - RN[d_2]\} \tag{2.8}$$

where

$$\begin{aligned} d_1 &= \frac{1}{\sigma_i\sqrt{T_i - t}}\left\{\ln\left(\frac{L_i(t)}{R}\right) + \frac{\sigma_i^2}{2}\cdot(T_i - t)\right\} \\ d_2 &= \frac{1}{\sigma_i\sqrt{T_i - t}}\left\{\ln\left(\frac{L_i(t)}{R}\right) - \frac{\sigma_i^2}{2}\cdot(T_i - t)\right\} \\ &= d_1 - \sigma_i\sqrt{T_i - t} \end{aligned}$$

where σ_i is the volatility of the interest rate of the period $(T_i - T_{i-1})$.

In this case,

$\alpha_i p_i(t) = (T_i - T_{i-1})e^{-y(T_i - t)}$. In Equation (2.8) there is the implicit assumption that the $L_i(t)$ are log-normally distributed in some sense. Our model will thus have to capture this property.

In the market, cap prices are quoted as implied volatilities, flat volatilities or as spot volatilities, also known as forward volatilities.

Let $t \leq T_0$ be fixed and R, the cap rate, be fixed. Assume that for each i there is a traded cap with resettlement dates $T_0, T_1, ..., T_N$.

Denote the corresponding observed cap market price by Cap_i^m. Then

$$Capl_i^m(t) = Cap_i^m(t) - Cap_{i-1}^m(t), \tag{2.9}$$

where $Cap_0^m(t) = 0$ and $Capl_1^m(t) = Cap_1^m(t)$. Clearly

$$Cap_i^m(t) \quad = \quad Capl_i^m(t) + Cap_{i-1}^m(t).$$

Thus

$$\begin{aligned} Cap_2^m(t) &= Capl_2^m(t) + Cap_1^m(t) \\ &= Capl_2^m(t) + Capl_1^m(t). \\ Cap_3^m(t) &= Capl_3^m(t) + Capl_2^m(t) + Capl_1^m(t). \end{aligned} \tag{2.10}$$

In general

$$Cap_i^m(t) = \sum_{k=1}^{i} Capl_k^m(t). \tag{2.11}$$

Definition 2.6 *Given market price data as above, the implied Black volatilities are defined as follows:*

(a) The implied Black flat volatilities $\bar{\sigma}_1, \ldots, \bar{\sigma}_N$ are defined as the solutions to the equations

$$Cap_i^m(t) = \sum_{k=1}^{i} Capl_k^B(t, \bar{\sigma}_i). \tag{2.12}$$

(b) The implied Black forward or spot volatilities $\bar{\sigma}_1, \ldots, \bar{\sigma}_N$ are defined as the solutions to the equations

$$Capl_i^m(t) = Capl_i^B(t, \bar{\sigma}_i). \tag{2.13}$$

The sequence of implied volatilities $\bar{\sigma}_1, \ldots, \bar{\sigma}_N$ (flat or spot) is called "volatility term structure".

Note that equation (2.12) can be written in matrix form as

$$\begin{pmatrix} Cap_1^m(t) \\ Cap_2^m(t) \\ \vdots \\ Cap_N^m(t) \end{pmatrix} = \begin{pmatrix} Capl_1^B(t, \bar{\sigma}_1) & 0 & \ldots & 0 \\ Capl_1^B(t, \bar{\sigma}_1) & \ldots & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ Capl_1^B(t, \bar{\sigma}_1) & \ldots & \ldots & 0 \\ Capl_1^B(t, \bar{\sigma}_1) & \ldots & \ldots & Capl_N^B(t, \bar{\sigma}_N) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{2.14}$$

2.2 The LIBOR market model: Risk Neutral Valuation Approach

The log-normal model for an asset price S at terminal time T is given by

$$S_T = S_0 e^{\sigma W_T + (\mu - \sigma^2/2)T} \tag{2.15}$$

where W_T is a normal random variable with mean 0 and variance T.

The parameter μ is the annual expected continuously compounded return earned by an investor. Investors who are not risk-averse usually require higher expected returns, i.e higher μ to induce them to take higher risks. Consequently, the value of μ should depend on the risk of the return of a stock, i.e, μ depends on the non-diversifiable risk.

The parameter σ is the stock price volatility and plays a crucial part in the valuation of most derivatives. The standard deviation of the proportional change in the stock price in a short time interval δt is $\sigma\sqrt{\delta t}$. The approximate standard deviation of the proportional change in stock price over a relatively long period of time T is $\sigma\sqrt{T}$. Hence, volatility can simply be interpreted as the standard deviation of the change in the stock price in a year. Taking the natural logarithm on both sides of the above equation we get

$$\ln S_T = \ln S_0 + \sigma W_T + (\mu - \frac{\sigma^2}{2})T. \tag{2.16}$$

The expression $\ln S_0 + (\mu - \frac{\sigma^2}{2})T$ is just the formula for a straight line and the random term σW_T jiggles the points about the line.

In continuous time, the solution to the stochastic differential equation

$$dS = \mu S dt + \sigma S dB \tag{2.17}$$

is the Geometric Brownian Motion (GBM)

$$S_t = S_0 \exp[\sigma B_t + (\mu - \sigma^2/2)t], \tag{2.18}$$

where B_t at each t is a normal random variable with mean 0 and variance t. Clearly,

$$\ln\left(\frac{S_t}{S_0}\right) = \sigma B_t + \left(\mu - \frac{\sigma^2}{2}\right) t. \tag{2.19}$$

The right hand side expression is a normal random variable with mean $(\mu - \frac{\sigma^2}{2})t$ and variance $\sigma^2 t$.

2.2.1 The LIBOR market model for Caps

Consider the theoretical no-arbitrage pricing of caps. The price $c_i(t)$ or $capl_i(t)$ of caplet number i is given by the standard risk neutral valuation formulae

$$\begin{aligned} Capl_i(t) &= \alpha_i E^Q \left[e^{-\int_t^{T_i} r(s)ds} \cdot \max[L_i(T_{i-1}) - R, 0] | \mathcal{F}_t \right], \ i = 1, \ldots, N \\ &= \alpha_i p_i(t) E^{T_i} \left[\max[L_i(T_{i-1}) - R, 0] | \mathcal{F}_t \right], \end{aligned}$$

where we use the forward measure Q^{T_i}, donoted by Q^i. Therefore E^{T_i} denotes the expectation with respect to the forward measure, or

$E^{Q^{T_i}}$. This follows from Theorem 1.13 with $P_\tau = Q^i$ and $E_\tau = E^{T_i}$.

Lemma 2.7 *For every $i = 1, \ldots, N$, the LIBOR process L_i is a martingale under the corresponding forward measure Q^i on the interval $[0, T_{i-1}]$.*

Proof

We wish to show that $E^{Q_i}[L_i(s)|\mathcal{F}_t] = L_i(t)$ for $0 \le t \le s \le T_{i-1}$.

$$\begin{aligned}
\alpha_i L_i(t) &= \frac{p_{i-1}(t)}{p_i(t)} - 1 \\
L_i(t) &= \frac{1}{\alpha_i}\left[\frac{p_{i-1}(t)}{p_i(t)} - 1\right] \\
E^{Q_i}[L_i(s)|\mathcal{F}_t] &= E^{Q_i}\left[\frac{1}{\alpha_i}\left\{\frac{p_{i-1}(s)}{p_i(s)} - 1\right\}|\mathcal{F}_t\right] \\
&= \frac{1}{\alpha_i}\left[E^{Q_i}\left\{\frac{p_{i-1}(s)(t)}{p_i(s)(t)} - 1\right\}|\mathcal{F}_t\right] \\
&= \frac{1}{\alpha_i}\left[\frac{p_{i-1}(t)}{p_i(t)} - 1\right] \\
&= L_i(t)
\end{aligned}$$

because p_i is the numeraire associated with Q^i and $p_{i-1}(s)$ is a price process over $[0, T_{i-1}]$. $\diamond$

From the above proof, we convince ourselves that L_i must have dynamics of the form

$$dL_i(t) = \Lambda(t)dW^i(t)$$

with $\Lambda(t)$ being of the form $\sigma_i(t)L_i(t)$ so that we can hope to capture the Black type formula. If for each i, the LIBOR rate $L_i(t)$ is log-normal under its own measure $Q^i = Q^{T_i}$, that is, if

$$\frac{dL_i(t)}{L_i(t)} = \sigma_i(t)dW^i(t), \tag{2.20}$$

where W^i is a Q^i Wiener process, then we say we have a LIBOR market model, because in this case we will have the simple distribution of $L_i(T_{i-1})$ and hence easily find $E^{T_i}[(L_i(T_{i-1}) - R)^+|\mathcal{F}_t]$. Now, assume we have a LIBOR market model. In the following section we show how to obtain the Black-76 formulae for caps and floors.

Since the equation in the expression (2.20) is just a GBM, we obtain

$$\begin{aligned} L_i(T) &= L_i(t)e^{\int_t^T \sigma_i(s)dW^i(s) - \frac{1}{2}\int_t^T ||\sigma_i(s)||^2 ds}. \\ \Longrightarrow \ln\left(\frac{L_i(T)}{L_i(t)}\right) &= \int_t^T \sigma_i(s)dW^i(s) - \frac{1}{2}\int_t^T ||\sigma_i(s)||^2 ds \end{aligned} \tag{2.21}$$

The right hand side of the above equation is a normal random variable with mean

$$m_i(t,T) = -\frac{1}{2}\int_t^T ||\sigma_i(s)||^2 ds \tag{2.22}$$

and variance

$$v_i^2(t,T) = \int_t^T ||\sigma_i(s)||^2 ds. \tag{2.23}$$

A few calculations give us the following proposition which gives us the price of a caplet with cap rate R.

Proposition 2.8

$$\begin{aligned} Capl_i(t) &= \alpha_i p_i(t) \left\{ L_i(t) N[d_1(t, T_{i-1})] - RN[d_2(t, T_{i-1})] \right\} \\ &= \alpha_i p_i(t) \left\{ L_i(t) N[d_1] - RN[d_2] \right\} \end{aligned} \tag{2.24}$$

where

$$\begin{aligned} d_1 &= \frac{1}{v_i(t, T_{i-1})} \left\{ \ln \left(\frac{L_i(t)}{R} \right) + \frac{1}{2} v_i^2(t, T_{i-1}) \right\} \\ d_2 &= d_1 - v_i(t, T_{i-1}). \end{aligned}$$

Proof Since

$$L_i(T) = L_i(t) \exp \left[\int_t^T \sigma_i(s) dW^i(s) - \frac{1}{2} \int_t^T ||\sigma_i(s)||^2 \right] ds, \tag{2.25}$$

the value of caplet i is given by

$$Capl_i(t) = \alpha_i p_i(t) E^{T_i} \left[\left(L_i(t) \exp \left\{ \int_t^T \sigma_i(s) dW^i(s) - \frac{1}{2} \int_t^T ||\sigma_i(s)||^2 ds \right\} - R \right)^+ |\mathcal{F}_t \right]. \tag{2.26}$$

Write $\int_t^T \sigma_i(s) dW^i(s)$ as $v_i x$ where $X \sim N(0,1)$. Thus

$$Capl_i(t) = \frac{\alpha_i p_i(t)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[L_i(t) \exp(v_i x - {v_i}^2/2) - R \right] e^{-x^2/2} dx$$

and

$$
\begin{aligned}
L_i(t)\exp(v_i x - {v_i}^2/2) - R > 0 \quad &\Leftrightarrow \quad \exp(v_i x - {v_i}^2/2) > \frac{R}{L_i(t)} \\
&\Leftrightarrow \quad v_i x - {v_i}^2/2 > \ln\left(\frac{R}{L_i(t)}\right) \\
&\Leftrightarrow \quad x > a = \frac{\ln(R/L_i(t)) + {v_i}^2/2}{v_i}.
\end{aligned}
$$

Hence

$$
Capl_i(t) = \frac{\alpha_i p_i(t)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[L_i(t)\exp(v_i x - {v_i}^2/2) - R\right] e^{-x^2/2} dx
$$

becomes

$$
\begin{aligned}
Capl_i(t) &= \frac{\alpha_i p_i(t)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[L_i(t)\exp(v_i x - {v_i}^2/2) - R\right] e^{-x^2/2} dx \\
&= \frac{\alpha_i p_i(t)}{\sqrt{2\pi}} \int_{a}^{\infty} L_i(t)\exp(v_i x - {v_i}^2/2) e^{-x^2/2} dx - \frac{\alpha_i p_i(t)}{\sqrt{2\pi}} R \int_{a}^{\infty} e^{-x^2/2} dx.
\end{aligned}
$$

Let

$$
\begin{aligned}
II &= -\frac{\alpha_i p_i(t)}{\sqrt{2\pi}} R \int_{a}^{\infty} e^{-x^2/2} dx \\
&= -\alpha_i p_i(t) R(1 - N(a)) \\
&= -\alpha_i p_i(t) R(N(-a)). \\
I &= \frac{\alpha_i p_i(t)}{\sqrt{2\pi}} \int_{a}^{\infty} L_i(t)\exp(v_i x - {v_i}^2/2) e^{-x^2/2} dx \\
&= \frac{\alpha_i p_i(t) L_i(t)}{\sqrt{2\pi}} \int_{a}^{\infty} e^{v_i x} e^{-{v_i}^2/2} e^{-x^2/2} dx \\
&= \frac{\alpha_i p_i(t) L_i(t)}{\sqrt{2\pi}} e^{-{v_i}^2/2} \int_{a}^{\infty} e^{v_i x - x^2/2} dx.
\end{aligned}
$$

Now completing the square,

$$
\begin{aligned}
v_i x - x^2/2 &= -\frac{1}{2}\left[x^2 - 2v_i x\right] \\
&= -\frac{1}{2}(x - v_i)^2 + v_i^2/2.
\end{aligned}
$$

Thus

$$
\begin{aligned}
I &= \frac{\alpha_i p_i(t) L_i(t)}{\sqrt{2\pi}} e^{-v_i{}^2/2} \int_a^\infty e^{-\frac{(x-v_i)^2}{2} + \frac{v_i^2}{2}} dx \\
&= \frac{\alpha_i p_i(t) L_i(t)}{\sqrt{2\pi}} e^{-v_i{}^2/2} \int_a^\infty e^{-\frac{(x-v_i)^2}{2}} e^{\frac{v_i^2}{2}} dx \\
&= \frac{\alpha_i p_i(t) L_i(t)}{\sqrt{2\pi}} e^{0} \int_a^\infty e^{-\frac{(x-v_i)^2}{2}} dx.
\end{aligned}
$$

Let $y = x - v_i$. Then $dy = dx$ and

$$
\begin{aligned}
I &= \frac{\alpha_i p_i(t) L_i(t)}{\sqrt{2\pi}} \int_{a-v_i}^\infty -e^{y^2/2} dy \\
&= \alpha_i p_i(t) L_i(t) \left[1 - N(a - v_i)\right] \\
&= \alpha_i p_i(t) L_i(t) N(-(a - v_i)).
\end{aligned}
$$

Hence

$$
\begin{aligned}
Capl_i(t) &= I + II \\
&= \alpha_i p_i(t) L_i(t) N(-(a - v_i)) - \alpha_i p_i(t) R(N(-a)) \\
&= \alpha_i p_i(t) \left[L_i(t) N(-(a - v_i)) - RN(-a)\right]. \qquad (2.27)
\end{aligned}
$$

Since

$$
a = \frac{\ln(R/L_i(t)) + v_i^2/2}{v_i},
$$

$$-a = \frac{\ln(L_i(t)/R) - v_i^2/2}{v_i}$$

and

$$\begin{aligned}
-(a - v_i) &= -\left[\frac{\ln(R/L_i(t)) + v_i^2/2}{v_i} - v_i\right] \\
&= -\left[\frac{\ln(R/L_i(t)) - v_i^2/2}{v_i}\right] \\
&= \frac{\ln(L_i(t)/R) + v_i^2/2}{v_i}.
\end{aligned}$$

Now letting $d_2 = -a$ and $d_1 = -(a - v_i)$, we have, as required that

$$Capl_i(t) = \alpha_i p_i(t) \left[L_i(t)N(d_1) - RN(d_2)\right].\diamond \tag{2.28}$$

The above proposition shows that;

$$\begin{aligned}
E^{T_i}[\max[L_i - R, 0]|\mathcal{F}_t] &= \text{call price in Black-Scholes framework} \\
& \text{in a world with zero short rate,} \\
&= \text{call price in Black-76 model framework.}
\end{aligned}$$

So Black's model can be justified via forward measures and Lemma 2.7, assuming L_i satisfies (2.20) for each i.

2.3 Floors: Definition and Market Practice

Definition 2.9 (Floorlet) *A floorlet is a European put option on the spot rate, the LIBOR rate in this case, at a fixed point in time.*

Definition 2.10 (Floor) *A floor is a portfolio of floorlets all having common exercise rate R which is the floor rate and with one floorlet for each period in a given interval of time.*

A floor provides investors with a guaranteed minimum rate of return. It protects investors against falling interest rates but allows them to benefit should interest rates firm.

If an investor has floating interest rate assets and wishes to protect against a decrease in short-term interest rates, buying a floor would allow the investor to ensure a minimum rate of return. If the market rate is higher than the floor rate, the option is not exercised and the investor invests at the higher market rate. In this way, the floor protects the investor against falling interest rates. In the South African market, settlement takes place on a quarterly basis against 3-month JIBAR and the purchase of the floor is paid either upfront or over the life of the floor.

Now consider a fixed set of increasing maturities $T_0,\ T_1,\ \ldots,\ T_N$ and define the tenor α_i by $\alpha_i = T_i - T_{i-1},\ i = 1, \ldots, N$ and the day-count factor by $1/\alpha_i$. Define the LIBOR forward rate contracted at

t for the period $[T_{i-1}, T_i]$ by

$$L_i(t) = \frac{1}{T_i - T_{i-1}} \left[\frac{p_{i-1}(t) - p_i(t)}{p_i(t)} \right]. \tag{2.29}$$

Then a floor with floor rate R and resettlement dates $T_0,\ T_1,\ \ldots,\ T_N$ is a contract which at time T_i gives the holder of the floor the amount

$$X_i = \alpha_i \cdot \max\left[R - L_i(t), 0\right]. \tag{2.30}$$

Recall that $L_i(t) \equiv L(t, T_i) \equiv L(t, T_{i-1}, T_i)$. Thus for $i = 1, ..., N$ floorlets we have the following pay-offs.

$$\begin{aligned}
X_1 &= \alpha_1 \cdot \max\left[R - L_1(t), 0\right] \\
X_2 &= \alpha_2 \cdot \max\left[R - L_2(t), 0\right] \\
\vdots & \qquad \ddots \qquad \vdots \\
X_N &= \alpha_N \cdot \max\left[R - L_N(t), 0\right].
\end{aligned} \tag{2.31}$$

The portfolio $\{X_1,\ X_2,\ ....,\ X_N\}$ is the floor.

Definition 2.11 (Black-76 Formula for floorlets) *The Black-76 formula for the floorlet whose pay-off is given by*

$$X_i = \alpha_i \cdot \max\left[R - L(T_{i-1}, T_i), 0\right] \tag{2.32}$$

is given by the expression

$$Floorl_i^B(t) = \alpha_i p_i(t)\{RN[-d_2] - L_i(t)N[-d_1]\} \tag{2.33}$$

where

$$\begin{aligned} d_1 &= \frac{1}{\sigma_i \sqrt{T_i - t}} \left\{ \ln\left(\frac{L_i(t)}{R}\right) + \frac{1}{2}\sigma_i^2 \cdot (T_i - t) \right\} \\ d_2 &= d_1 - \sigma_1 \sqrt{T_i - t} \end{aligned}$$

where σ_i is the volatility of the interest rate of the period $(t_i - t_{i-1})$.

Analogous to caps, if we denote the corresponding observed market price by $Floorl_i^m(t)$, then

$$Floorl_i^m(t) = Floor_i^m(t) - Floor_{i-1}^m(t) \tag{2.34}$$

where $Floor_0^m(t) = 0$ and $Floorl_1^m(t) = Floor_1^m(t)$. In general,

$$Floor_i^m(t) = \sum_{k=1}^{i} Floorl_k^m(t). \tag{2.35}$$

The implied Black flat volatilities $\bar{\sigma}_1, \ldots, \bar{\sigma}_N$ are defined as the solutions to the equations

$$Floor_i^m(t) = \sum_{k=1}^{i} Floorl_k^m(t, \bar{\sigma}_i) \tag{2.36}$$

which are equivalent to

$$\begin{pmatrix} Floor_1^m(t) \\ Floor_2^m(t) \\ \vdots \\ Floor_N^m(t) \end{pmatrix} = \begin{pmatrix} Floorl_1^B(t,\bar{\sigma}_1) & 0 & \dots & 0 \\ Floorl_1^B(t,\bar{\sigma}_1) & Floorl_2^B(t,\bar{\sigma}_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ Floorl_1^B(t,\bar{\sigma}_1) & \dots & \dots & 0 \\ Floorl_1^B(t,\bar{\sigma}_1) & \dots & \dots & Floorl_N^B(t,\bar{\sigma}_N) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \tag{2.37}$$

2.3.1 Floors: The LIBOR market model

With all the assumptions from the section on the pricing of caps in the LIBOR market, we state without proof the following proposition. (The proof is similar to that of Proposition 2.9.)

Proposition 2.12 (Price of a floorlet)

$$Floorl_i(t) = \alpha_i p(t) \{RN(-d_2) - L_i(t)N(-d_1)\}, \tag{2.38}$$

where

$$\begin{aligned} d_1 &= \frac{1}{v_i(t,T_{i-1})}\left[\ln(\frac{L_i(t)}{R}) + \frac{1}{2}v_i^2(t,T_{i-1})\right], \\ d_2 &= d_1 - v_i(t,T_{i-1}). \end{aligned}$$

2.4 Terminal Measure Dynamics and Existence of LIBOR market model

Consider deterministic volatilities $\sigma_1, \ldots, \sigma_N$ and for all the LIBOR rates L_i, choose Q^N as the common measure. Also consider a deterministic function μ_i such that

$$dL_i(t) = L_i(t)\mu_i(t, L(t))dt + L_i(t)\sigma_i(t)dW^N(t) \tag{2.39}$$

where $L(t) = [L_1(t), \ldots, L_N(t)]'$. We wish to show that for some suitable choice of μ_i, the Q^N dynamics above will imply Q^i dynamics of the form seen in the previous chapter.

Denoted by η: $\eta_i^j(t) = \frac{dQ^j}{dQ^i}$ the likelihood process on $\mathcal{F}_t$ with $i, j = 1, \ldots, N$.

Then the Radon-Nikodym derivative η_i^j is given by

$$\eta_i^j(t) = \frac{p_i(0)}{p_j(0)}\frac{p_j(t)}{p_i(t)}. \tag{2.40}$$

In particular, for $j = i - 1$,

$$\begin{aligned}
\eta_i^{i-1}(t) &= \frac{p_i(0)}{p_{i-1}(0)}\frac{p_{i-1}(t)}{p_i(t)} \\
&= a_i\frac{p_{i-1}(t)}{p_i(t)}
\end{aligned}$$

$$= \quad a_i[1+\alpha_i L_i(t)], \tag{2.41}$$

where $a_i = \frac{p_i(0)}{p_{i-1}(0)}$.

Then the η_i^{i-1} dynamics under Q^i are given by

$$d\eta_i^{i-1}(t) = a_i \alpha_i dL_i(t). \tag{2.42}$$

RECALL: $L_i(t) = \frac{1}{\alpha_i}\frac{p_{i-1}(t)-p_i(t)}{p_i(t)} = \frac{1}{\alpha_i}\left[\frac{p_{i-1}(t)}{p_i(t)} - 1\right]$.

Assuming the L_i - dynamics are as in the previous chapter, i.e,

$$dL_i(t) = L_i(t)\sigma_i(t)dW^i(t), \tag{2.43}$$

and using RECALL we get the $\eta_i^{i-1}(t)$ dynamics as

$$\begin{aligned}
d\eta_i^{i-1}(t) &= a_i\alpha_i L_i(t)\sigma_i(t)dW^i(t) \\
&= a_i\alpha_i\left[\frac{1}{\alpha_i}\left(\frac{p_{i-1}(t)}{p_i(t)} - 1\right)\right]\sigma_i(t)dW^i(t) \\
&= a_i\left[\frac{p_{i-1}(t)}{p_i(t)} - 1\right]\sigma_i(t)dW^i(t) \\
&= \eta_i^{i-1}(t)a_i\frac{1}{\eta_i^{i-1}(t)}\left[\frac{p_{i-1}(t)}{p_i(t)} - 1\right]\sigma_i(t)dW^i(t) \\
&= \eta_i^{i-1}(t)a_i\frac{1}{a_i(1+\alpha_i L_i(t))}\left[\frac{p_{i-1}(t)}{p_i(t)} - 1\right]\sigma_i(t)dW^i(t) \\
&= \eta_i^{i-1}(t)\frac{1}{1+\alpha_i L_i(t)}\alpha_i L_i(t)\sigma_i(t)dW^i(t) \\
&= \eta_i^{i-1}(t)\frac{\alpha_i L_i(t)}{1+\alpha_i L_i(t)}\sigma_i(t)dW^i(t).
\end{aligned} \tag{2.44}$$

Thus the Girsanov kernel for η_i^{i-1} is given by

$$\frac{\alpha_i L_i(t)}{1+\alpha_i L_i(t)}\sigma_i{}^\star(t) \tag{2.45}$$

where W^i is k-dimensional and each σ_i is a k-dimensional vector $\sigma_i = [{\sigma_i}^1, {\sigma_i}^2, \ldots, {\sigma_i}^k]$. ${\sigma_i}^\star$ is the transpose of σ_i

From the Girsanov theorem we have

$$dW^i(t) = \frac{\alpha_i L_i(t)}{1+\alpha_i L_i(t)}{\sigma_i}^\star(t)dt + dW^{i-1}(t). \tag{2.46}$$

Inductively ,

$$dW^i(t) = -\sum_{k=i+1}^{N} \frac{\alpha_k L_k(t)}{1+\alpha_k L_k(t)}{\sigma_k}^\star(t)dt + dW^N(t), \tag{2.47}$$

and

$$\begin{aligned} dL_i(t) &= L_i(t)\sigma_i(t)dW^i(t) \\ &= L_i(t)\sigma_i(t)\left[-\sum_{k=i+1}^{N} \frac{\alpha_k L_k(t)}{1+\alpha_k L_k(t)}{\sigma_k}^\star(t)dt + dW^N(t)\right] \\ &= -L_i(t)\left(\sum_{k=i+1}^{N} \frac{\alpha_k L_k(t)}{1+\alpha_k L_k(t)}{\sigma_k}^\star(t)\sigma_i(t)dt\right) + L_i(t)\sigma_i(t)dW^N(t), \end{aligned} \tag{2.48}$$

which is the following proposition.

Proposition 2.13 *Let $\sigma_1, \ldots, \sigma_N$, be a given volatility structure where each σ_i is assumed to be bounded. Also, consider a probability measure Q^N and a standard Q^N-Wiener process W^N and define the processes $L_1, \ldots, L_N$ by*

$$dL_i(t) = -L_i(t)\left(\sum_{k=i+1}^{N} \frac{\alpha_k L_k(t)}{1+\alpha_k L_k(t)}{\sigma_k}^\star(t)\sigma_i(t)dt\right) + L_i(t)\sigma_i(t)dW^N(t), \tag{2.49}$$

with $i = 1, \ldots, N$. Then,

(1) the Q^i - dynamics of L_i are given by equation (2.20).

(2) there exists a LIBOR model with the given volatility structure.

Proof

(1) We prove that the Q^i - dynamics of L_i are given by equation (2.20). We will use the convention that $\sum_{N+1}^{N}(\cdots) = 0$. For $i = N$ we see that

$$\begin{aligned} dL_N(t) &= -L_N(t)\underbrace{\left(\sum_{k=N+1}^{N} \frac{\alpha_k L_k(t)}{1+\alpha_k L_k(t)} \sigma_k{}^{\star}(t)\sigma_N(t)dt\right)}_{=0} + L_N(t)\sigma_N(t)dW^N(t), \\ &= L_N(t)\sigma_N(t)dW^N(t). \end{aligned}$$

which is just equation (2.20)!

(2) Now we show that there exists a LIBOR model with the given volatility structure, i.e, that there exists a solution for equation 2.49. We prove by mathematical induction.

$dL_N(t) = L_N(t)\sigma_N(t)dW^N(t)$ is just a GBM. Since, by assumption, σ_N is bounded, a solution does exist. Assume that the solution exists for $k = i+1, \ldots, N$. We can then write the i-th component of the equation as

$$dL_i(t) = L_i(t)\mu_i\left[t, L_{i+1}(t), \ldots, L_N(t)\right]dt + L_i(t)\sigma_i(t)dW^N(t), \tag{2.50}$$

where μ_i only depends on L_k for $k = i+1, \ldots, N$ and not on L_i. Denote $(L_{i+1}, \ldots, L_N)$ by L_{i+1}^N. Then we have the explicit solution

$$\begin{aligned} L_i(t) &= L_i(0) \exp\left\{ \int_0^t \left(\mu_i[s, L_{i+1}]^N(s)] - \frac{1}{2}||\sigma_i(s)||^2 \right) ds \right\} \times \\ &\times \exp\left\{ \int_0^t \mu_i[s, L_{i+1}^N(s)] dW^N(s) \right\} . \diamond \end{aligned}$$

2.5 Interest Rate Collars: Market Practice

[27] The buyer of an interest rate collar purchases an interest rate cap while selling a floor indexed to the same interest rate. Borrowers with variable-rate loans buy collars to limit effective borrowing rates to a range of interest rates between some maximum, determined by the cap rate, and a minimum, which is fixed by the floor strike price; hence the term "collar". Although buying a collar limits a borrower's ability to benefit from a significant decline in market interest rates, it has the advantage of being less expensive than buying a cap alone because the borrower earns premium income from the sale of the floor that offsets the cost of the cap. A zero-cost collar results when the premium earned by selling a floor exactly offsets the cap premium.

The amount of the payment (pay-off) due to or owed by a buyer of an interest rate collar is determined by the expression

$$N\left(\frac{dt}{360}\right)\left[\max(r-r_c,0)-\max(r_f-r,0)\right], \qquad (2.51)$$

where

N - is the notional principal of the agreement,

r_c - is the cap rate,

r_f - is the floor rate,

dt - is the term of the index days, i.e number of days,

r - is the index interest rate.

Note that depending on the usual conventions, 365 is also used instead of 360.

If the index interest rate r is less than the floor rate r_f on the interest rate reset date, the floor is in-the-money and the collar buyer (who has sold a floor) must pay the collar counter-party an amount equal to $N\frac{dt}{360}(r_f - r)$. When $r_f < r < r_c$, both the floor and the cap are out-of-the-money and no payments are exchanged. Finally, when the index is above the cap rate the cap is in-the-money and the buyer receives $N\frac{dt}{360}(r - r_c)$.

A special case is the zero-cost collar that results from the simultaneous purchase of a one-period cap and sale of a one-period floor when the cap and floor rates are equal. In this case the combined transaction replicates the pay-off of a FRA with a forward interest rate equal to the cap/floor rate. This result is a consequence of a property of option prices known as put-call parity.

2.5.1 Pricing collars in the LIBOR market models

Consider a fixed set of increasing maturities $T_0, T_1, \ldots, T_n$ and define $\alpha_i = T_i - T_{i-1}$ as the tenor and $1/\alpha_i$ as the day-count factor.

Definition 2.14 (Collar) *A collar, with resettlement dates $T_0, T_1, \ldots, T_N$, is a combination of a cap with cap rate R_c and a floor with floor rate R_f. It pays off the amount*

$$X_i = \alpha_i \left[\max(L_i(T_{i-1}) - R_c, 0) - \max(R_f - L_i(T_{i-1}), 0)\right]. \quad (2.52)$$

But, since

$$\begin{aligned} X_i &= \alpha_i \left[\max(L_i(T_{i-1}) - R_c, 0) - \max(R_f - L_i(T_{i-1}), 0)\right] \\ &= \alpha_i \left[\max(L_i(T_{i-1}) - R_c, 0)\right] - \alpha_i[\max(R_f - L_i(T_{i-1}), 0)], \end{aligned}$$

the price of a collar is given by the expression

$$Collar_i^B(t) = \alpha_i p_i(t) \left\{[L_i(t)N(d_1) - R_c N(d_2)] - [R_f N(-d_2{}') - L_i(t)N(-d_1{}')]\right\} \quad (2.53)$$

where

$$
\begin{aligned}
d_1 &= \frac{1}{v_i(t,T_{i-1})}\left[\ln(\frac{L_i(t)}{R_c}) + \frac{1}{2}v_i^2(t,T_{i-1})\right],\\
d_2 &= d_1 - v_i(t,T_{i-1}),\\
{d_1}' &= \frac{1}{{v_i}'(t,T_{i-1})}\left[\ln(\frac{L_i(t)}{R_f}) + \frac{1}{2}{v_i}'^2(t,T_{i-1})\right],\\
{d_2}' &= {d_1}' - {v_i}'(t,T_{i-1}).
\end{aligned}
$$

Note that the above equation for the price of a collar can be written as

$$
Collar_i^B(t) = \alpha_i p_i(t)\left\{L_i(t)[N(d_1) + N(-{d_1}')] - [R_c + R_f][N(d_2) - N(-{d_2}')]\right\}. \tag{2.54}
$$

Chapter 3

The Swap Market Model

3.1 Swaps

Definition 3.1 *A swap is an agreement between two parties to exchange cash-flows in the future, at some agreed dates.*

The most common type of swap is a "plain vanilla" interest rate swap. Here company B agrees to pay company A cash flows equal to interest at a pre-determined fixed rate on a notional principal (it is not exchanged but used only for the calculation of interest payments) for a number of years. At the same time company A agrees to pay company B cash-flows equal to interest at a floating rate, which, in many interest rate swap agreements, is the LIBOR (or the

JIBAR in the South African market[Chapter 4]). Exchanging the same amount makes no sense, hence the principal is not exchanged.

3.1.1 Valuation of Interest rate swaps: Market Practice

When swaps and other over-the-counter derivatives are valued, the cash-flows are usually discounted using LIBOR zero-coupon interest rates. This is because LIBOR is the cost of funds for a financial institution.

Relationship of swaps to bonds

A swap is the same as an agreement in which

1. Company B has lent company A a certain amount (not principal) at the x-month LIBOR rate.

2. Company A has lent company B the same amount at a fixed rate per annum.

The value of the money to B is therefore the difference between the values of the two bonds. Define

B_{fix} - time 0 value of fixed-rate bond underlying the swap,

B_{float} - time 0 value of floating-rate bond underlying the swap. Then

V, the value of the swap to company B is

$$V_{swap} = B_{float} - B_{fix}. \tag{3.1}$$

If all interest and principal are realized at the end of the period, say n years, then the rate involved is called an n-year zero rate, also known as zero-coupon rate or n-year spot rate. Define:

t_i: time when ith payments are exchanged, $i = 1, \ldots, n$,

L: notional principal in swap agreement,

$L_i = L_i(0) = L(0, t_i)$: LIBOR zero- rate for a maturity t_i,

K: fixed payment made on each payment date.

Then,

$$B_{fix} = \sum_{i=1}^{n} K e^{-L_i t_i} + L e^{-L_n t_n}. \tag{3.2}$$

For the floating rate bond, immediately after a payment date, we have $B_{float} = L$ because this is now identical to a newly issued floating rate bond. But, immediately before the next payment date, we have $B_{float} = L$ plus floating rate payment, say K^*, to be paid on the next payment date. Today's swap value is its value before tomorrow's payment discounted at the LIBOR rate L_1 for time t_1, i.e,

$$B_{float} = (L + K^*) e^{-L_1 t_1} \tag{3.3}$$

where K^* is the floating-rate payment already known.

Substituting the two equations into V_{swap} we get

$$V_{swap} = -\left(\sum_{i=1}^{n} Ke^{-L_i t_i} + Le^{-L_n t_n}\right) + (L + K^*)e^{-L_1 t_1}. \qquad (3.4)$$

The value of the swap to A will be negative. K^* is, in precise form, calculated taking into account the accrual day-count convention (out of 365 or 360 days).

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The set of floating rate payments is called the floating leg while that of fixed rate payments is called the fixed leg.

Receiver swap: in this case the holder of a receiver swap receives the fixed leg and pays the floating leg.

Payer swap: the holder of this one pays the fixed leg and receives the floating leg.

3.1.2 General Theory of Swaps

Now consider resettlements dates $T_0, T_1, \ldots, T_N$; $\alpha_i = T_i - T_{i-1}$.

Definition 3.2 *The payments in a $T_n \times (T_N - T_n)$ payer swap are as follows:*

- Payments will be made and received at $T_{n+1}, T_{n+2}, \ldots, T_N$.

- For every elementary period $[T_i, T_{i+1}], i = 1, \ldots, N-1$, *the LIBOR rate* $L_{i+1}(T_i)$ *is set at time* T_i *and the floating leg* $\alpha_{i+1}L_{i+1}(T_i)$ *is received at* T_{i+1}. *We assume a notional principal of* $L \equiv 1$.

- For the same period the fixed leg $\alpha_{i+1}K$ *is paid at* T_{i+1}, *where* K *is a fixed rate (swap rate). This* K *is not quite the same as the one* *on page 63.*

If an amount of

$$\alpha_{i+1}L_{i+1}(T_i) = \frac{p(T_i, T_i) - p(T_i, T_{i+1})}{p(T_i, T_{i+1})}$$

is received at time T_{i+1}, then $\alpha_{i+1}L_{i+1}(T_i)p(t, T_{i+1})$ is received at time T_i. But this is just $p(T_i, T_i) - p(T_i, T_{i+1})$. If payoff of this contract at time T_i is $p(T_i, T_i) - p(T_i, T_{i+1})$, then because of no-arbitrage, the value of the floating payment at time t is given by the expression

$$p(t, T_i) - p(t, T_{i+1}). \tag{3.5}$$

Hence, the total value of the floating side at time t for $t \leq T_n$ is

$$\begin{aligned} p(t, T_n) - p(t, T_{n+1}) + p(t, T_{n+1}) - p(t, T_{n+2}) &+ \cdots + p(t, T_{N-1}) - p(t, T_N) \\ &= \sum_{i=n}^{N-1} [p(t, T_i) - p(t, T_{i+1})] \end{aligned}$$

$$
\begin{aligned}
&= p(t,T_n) - p(t,T_N) \\
&= p_n(t) - p_N(t).
\end{aligned}
$$

The total value on the fixed side is

$$
\begin{aligned}
\sum_{i=n}^{N-1} p(t,T_{i+1})\alpha_{i+1}K &= K\sum_{i=n+1}^{N}\alpha_i p(t,T_i) \\
&= K\sum_{i=n+1}^{N}\alpha_i p_i(t).
\end{aligned}
$$

The net value $PS_n^N(t,K)$ of the $T_n \times (T_N - T_n)$ payer swap at time $t < T_n$ is

$$
\begin{aligned}
B_{float} - B_{fix} &= p(t,T_n) - p(t,T_N) - K\sum_{i=n+1}^{N}\alpha_i p_i(t) \\
\text{i.e } PS_n^N(t,K) &= p_n(t) - p_N(t) - K\sum_{i=n+1}^{N}\alpha_i p_i(t). \qquad (3.6)
\end{aligned}
$$

But

$$
\begin{aligned}
p_n(t) - p_N(t) - K\sum_{i=n+1}^{N}\alpha_i p_i(t) &= 0 \\
\Leftrightarrow K &= \frac{p_n(t) - p_N(t)}{\sum_{i=n+1}^{N}\alpha_i p_i(t)}.
\end{aligned}
$$

Definition 3.3 *The par or forward swap rate $R_n^N(t)$ of the $T_n \times (T_N - T_n)$ swap is the value of K for which $PS_n^N(t,K) = 0$, i.e,*

$$
K = R_n^N(t,K) = \frac{p_n(t) - p_N(t)}{\sum_{i=n+1}^{N}\alpha_i p_i(t)}. \qquad (3.7)
$$

Definition 3.4 *For each pair n, k with $n < k$, the process in the* *denominator of the above equation, $S_n^k(t)$, defined by*

$$S_n^k(t) = S^k(t, T_n) = \sum_{i=n+1}^{k} \alpha_i p_i(t) \tag{3.8}$$

is known as the accrual factor or as the present value of a basis point.

Note that ${S_n}^k(t)$ represents the value at time t of a portfolio of bonds with different maturities.

It is clear that

$$R_n^N(t) = \frac{p_n(t) - p_N(t)}{S_n^N(t)},\ 0 \leq t \leq T_n. \tag{3.9}$$

Hence, the arbitrage-free price of a payer swap with swap rate K is

$$\begin{aligned} PS_n^N(t, K) &= p_n(t) - p_N(t) - K \sum_{i=n+1}^{N} \alpha_i p_i(t) \\ &= p_n(t) - p_N(t) - KS_n^N(t) \\ &= R_n^N(t, K) \sum_{i=n+1}^{N} \alpha_i p_i(t) - KS_n^N(t) \\ &= R_n^N(t, K) S_n^N(t) - KS_n^N(t) \\ &= \left[R_n^N(t, K) - K\right] S_n^N(t) \\ &= \left[R_n^N(t) - K\right] S_n^N(t) \end{aligned} \tag{3.10}$$

Equally, the price of a receiver swap is given by

$RS_n^N(t) = \left[K - R_n^N(t)\right] S_n^N(t).$

3.2 Swaptions: Definition and Market Practice

Definition 3.5 *A $T_n \times (T_N - T_n)$ payer swaption with strike K is a contract which at the exercise date T_n gives the holder the right but not the obligation to enter into a $T_n \times (T_N - T_n)$ swap with fixed swap rate K.*

We see from the definition that a payer swaption is a contingent T_n-claim that pays

$$\begin{aligned} X_n^N &= \max\left[PS_n^N(T_n, K), 0\right] \\ &= \max\left[\left(R_n^N(T_n, K) - K\right) S_n^N(T_n), 0\right] \\ &= S_n^N(T_n) \max\left[(R_n^N(T_n) - K), 0\right] \end{aligned} \tag{3.11}$$

which is a call option on R_n^N with strike K. Hence,

Definition 3.6 (Black's formula for swaptions) *The Black-76 formula for a $T_n \times (T_N - T_n)$ payer swaption with strike K is defined as*

$$PS_n^N(t) = S_n^N(t) \left\{R_n^N(t) N[d_1] - K N[d_2]\right\}, \tag{3.12}$$

where

$$d_1 = \frac{1}{\sigma_{n,N}\sqrt{T_n - t}} \left[\ln\left(\frac{R_n^N(t)}{K}\right) + \frac{1}{2}\sigma_{n,N}^2 (T_n - t)\right]$$

$$d_2 = d_1 - \sigma_{n,N}\sqrt{T_n - t}.$$

The constant $\sigma_{n,N}$ is known as the Black volatility. Given a market price for the swaption, the Black volatility implied by the Black formula is referred to as the implied Black volatility.

The task at hand is to build an arbitrage-free model with the property that the theoretical prices derived within the model has the structure of the Black formula in the above definition.

3.3 The Swap Market Models

Lemma 3.7 *Denote the martingale measure for the numeraire $S_n^k(t)$ by Q_n^k. Then the forward swap rate R_n^k is a Q_n^k - martingale.*

Proof 3.8 *We are required to prove that $E\left(R_n^k(s)|\mathcal{F}_t\right) = R_n^k(t)$.*

$$\begin{aligned} E^{Q_n^k}\left(R_n^k(s)|\mathcal{F}_t\right) &= E^{Q_n^k}\left[\frac{p_n(s) - p_k(s)}{S_n^k(s)}|\mathcal{F}_t\right], 0 \leq t \leq s \\ &= \frac{p_n(t) - p_k(t)}{S_n^k(t)} \\ &= R_n^k(t) \end{aligned}$$

since R_n^k is the value of a self-financing portfolio (a long T_n bond and a short T_k bond), divided by the value of the self-financing portfolio $S_n^k(t)$.

Definition 3.9 *Consider resettlement dates $T_0, \dots, T_N$, with $0 \leq n < k \leq N$. Furthermore, consider a deterministic function of time $\sigma_{n,k}(t)$. A swap market model with volatilities $\sigma_{n,k}(t)$ is specified by assuming that the par swap rates have dynamics of the form*

$$dR_n^k(t) = R_n^k(t)\sigma_{n,k}(t)dW_n^k(t), \tag{3.13}$$

where $W_n^k(t)$ is Wiener under Q_n^k.

3.3.1 Pricing Swaptions in the Swap Market Model

The swap market model price of a $T_n \times (T_N - T_n)$ swaption is

$$PSN_n^N(t) = S_n^N(t)E^{n,N}\left[\max\left[R_n^N(T_n) - K, 0\right]|\mathcal{F}_t\right], 0 \leq t \leq T_n. \tag{3.14}$$

Since the equation in (3.13) describing the dynamics of R_n^N is a GBM, then

$$R_n^N(T_n) = R_n^N(t)e^{\int_t^{T_n} \sigma_{n,N}(t)dW_n^k(s) - \frac{1}{2}\int_t^{T_n} ||\sigma_{n,N}(s)||^2 ds}. \tag{3.15}$$

And, since $\sigma_{n,N}$ is deterministic, then conditional on $\mathcal{F}_t$, the process $R_n^N(T_n)$ is log-normal, that is we can write

$$R_n^N(T_n) = R_n^N(t)e^{Y_n^N(t,T_n)}, \tag{3.16}$$

where $Y_n^N(t, T_n)$ is normally distributed with expected value

$$m_n^N(t, T_n) = -\frac{1}{2}\int_t^{T_n} ||\sigma_{n,N}(s)||^2 ds, \tag{3.17}$$

and variance

$$v_n^{N^2}(t, T_n) = \int_t^{T_n} ||\sigma_{n,N}(s)||^2 ds. \tag{3.18}$$

These results give rise to the following swaption pricing formula.

Proposition 3.10 *In the swap market model, the $T_n \times (T_N - T_n)$ payer swaption price with strike K is given by*

$$PSN_n^N(t) = S_n^N(t)\left\{R_n^N(t)N[d_1] - KN[d_2]\right\}, \tag{3.19}$$

where

$$\begin{aligned} d_1 &= \frac{1}{v_n^N\sqrt{T_n - t}}\left[\ln\left(\frac{R_n^N(t)}{K}\right) + \frac{1}{2}v_n^{N^2}\right] \\ d_2 &= d_1 - \sigma_{n,N} \end{aligned}$$

which is of the form of the Black-76 formula in Definition 3.6.

Equation (3.19) shows that the numeraire for pricing swaptions is $S_n^N(t)$, whereas the numeraire for pricing caplets (floorlets) is $\alpha_i p_i(t)$.

Proof

The pay-off of a payer swaption is given by

$$PSN_n^N(t) = S_n^N(t)E^{n,N}\left[\max[R_n^N(T_n) - K, 0]|\mathcal{F}_t\right]. \tag{3.20}$$

But since $R_n^N(t)$ is a GMB,

$$\begin{aligned} R_n^N(T_n) &= R_n^N(t)\exp\left\{\int_t^{T_n}\sigma_{n,N}(t)dW_n^k(s) - \frac{1}{2}\int_t^{T_n}||\sigma_{n,N}(s)||^2 ds\right\} \\ &= R_n^N(t)e^{Y_n^N(t,T_n)}, \end{aligned} \tag{3.21}$$

by the deterministic character of $\sigma_{n,N}$ and the log-normality of $R_n^N(T_n)$.

Here

$Y_n^N(t,T_n) \sim N(\mu,\sigma) = N\left(-\frac{1}{2}\int_t^{T_n}||\sigma_{n,N}(s)||^2 ds, (\int_t^{T_n}||\sigma_{n,N}(s)||^2 ds)^{1/2}\right).$

Now letting

$$\begin{aligned} m_n^N(t,T_n) &= -\frac{1}{2}\int_t^{T_n}||\sigma_{n,N}(s)||^2 ds, \\ {v_n^N}^2(t,T_n) &= \int_t^{T_n}||\sigma_{n,N}(s)||^2 ds, \end{aligned}$$

the value of the payer swaption is given by

$$PSN_n^N(t) = S_n^N(t)E^{n,N}\max\left[[R_n^N(T_n) - K, 0]^+\right]. \tag{3.22}$$

Write $\int_t^{T_n}\sigma_{n,N}(t)dW_n^N(t)$ as $v_n^N x$ where $X \sim N(0,1)$. Then

$$PSN_n^N(t) = \frac{S_n^N(t)}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left\{R_n^N(t)\exp\left(v_n^N x - \frac{{v_n^N}^2}{2}\right) - K\right\}^+ e^{-x^2/2}dx \tag{3.23}$$

and

$$R_n^N(t)\exp\left(v_n^N x - \frac{{\sum_n^N}^2}{2}\right) - K \quad = \quad 0$$

$$\Leftrightarrow \exp\left(\sum_n^N x - \frac{\sum_n^{N\,2}}{2}\right) = \frac{K}{R_n^N}.$$

Solve the following equation for x:

$$\begin{aligned}
\exp\left(v_n^N x - \frac{v_n^{N\,2}}{2}\right) &= \frac{K}{R_n^N} \\
v_n^N x - \frac{v_n^{N\,2}}{2} &= \ln\left(\frac{K}{R_n^N}\right) \\
x = a &= \frac{\ln\left(\frac{K}{R_n^N}\right) + \frac{v_n^{N\,2}}{2}}{v_n^N}. \qquad (3.24)
\end{aligned}$$

Thus

$$\begin{aligned}
PSN_n^N &= \frac{S_n^N(t)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ R_n^N(t) \exp\left(v_n^N x - \frac{v_n^{N\,2}}{2}\right) - K \right\}^+ e^{-x^2/2} dx \\
&= \frac{S_n^N(t)}{\sqrt{2\pi}} \int_a^{\infty} R_n^N(t) e^{v_n^N x} e^{-\frac{v_n^{N\,2}}{2}} e^{-x^2/2} dx - \frac{S_n^N(t)}{\sqrt{2\pi}} \int_a^{\infty} K e^{-x^2/2} dx.
\end{aligned}$$

Let

$$\begin{aligned}
II &= -\frac{S_n^N(t)}{\sqrt{2\pi}} \int_a^{\infty} K e^{-x^2/2} dx \\
&= -S_n^N(t) K \left[\frac{1}{\sqrt{2\pi}} \int_a^{\infty} e^{-x^2/2} dx\right] \\
&= -S_n^N(t) K (1 - N(a)).
\end{aligned}$$

And let

$$\begin{aligned}
I &= \frac{S_n^N(t)}{\sqrt{2\pi}} \int_a^{\infty} R_n^N(t) e^{v_n^N x} e^{-\frac{v_n^{N\,2}}{2}} e^{-x^2/2} dx \\
&= \frac{S_n^N(t) R_n^N(t) e^{-\frac{v_n^{N\,2}}{2}}}{\sqrt{2\pi}} \int_a^{\infty} e^{v_n^N x - x^2/2} dx.
\end{aligned}$$

But

$$\begin{aligned}
v_n^N x - x^2/2 &= -\frac{1}{2}\left[x^2 - 2v_n^N x\right] \\
&= -\frac{1}{2}\left[x^2 - 2v_n^N x + (-v_n^N)^2 - (-v_n^N)^2\right] \\
&= -\frac{1}{2}\left(x - v_n^N\right)^2 + \frac{{v_n^N}^2}{2}. \\
I &= \frac{S_n^N(t)R_n^N(t)e^{-\frac{{v_n^N}^2}{2}}}{\sqrt{2\pi}} \int_a^\infty e^{-\frac{1}{2}\left(x-v_n^N\right)^2+\frac{{v_n^N}^2}{2}}dx \\
&= \frac{S_n^N(t)R_n^N(t)\cdot 1}{\sqrt{2\pi}} \int_a^\infty e^{-\frac{1}{2}(x-v_n^N)^2}dx. \qquad (3.25)
\end{aligned}$$

Let $y = x - v_n^N$. Then $dy = dx$ and $a - v_n^N \leq y < \infty$ as $a \leq x < \infty$. Hence

$$\begin{aligned}
I &= \frac{S_n^N(t)R_n^N(t)e^{-\frac{{v_n^N}^2}{2}}}{\sqrt{2\pi}} \int_{a-v_n^N}^\infty e^{-\frac{y^2}{2}}e^{\frac{{v_n^N}^2}{2}}dy \\
&= S_n^N(t)R_n^N(t)\left[\frac{1}{\sqrt{2\pi}}\int_{a-v_n^N}^\infty e^{-\frac{y^2}{2}}dy\right] \\
&= S_n^N(t)R_n^N(t)\left[1 - N(a - v_n^N)\right] \\
&= S_n^N(t)R_n^N(t)N\left(-(a - v_n^N)\right). \\
PSN_n^N(t) &= I + II \\
&= S_n^N(t)R_n^N(t)N\left(-(a - v_n^N)\right) + S_n^N(t)K(1 - N(a)) \\
&= S_n^N(t)\left[R_n^N(t)N\left(-(a - v_n^N)\right) + K(1 - N(a))\right].
\end{aligned}$$

Now if we let $d_1 = -(a - v_n^N)$ and $d_2 = -a = \frac{\ln(R_n^N/K) - {v_n^N}^2}{v_n^N}$, we get that

$$PSN_n^N = S_n^N(t)\left[R_n^N(t)N(d_1) - KN(d_2)\right].\diamond \qquad (3.26)$$

In the same manner we obtain that the price of a receiver swaption is given by

$$RSN_n^N(t) = S_n^N(t)\left[KN(-d_2) - R_n^N(t)N(-d_1)\right]. \qquad (3.27)$$

3.4 Compatibility of LIBOR and Swap market models

Consider an elementary period $[T_i, T_{i+1}]$. Then

$$\begin{aligned}
R_i^{\,i+1}(t) &= \frac{p_i(t) - p_{i+1}(t)}{S_i^{\,i+1}(t)} \\
&= \frac{p_i(t) - p_{i+1}(t)}{\alpha_{i+1} p_{i+1}} \\
&= \frac{p_i(t) - p_{i+1}(t)}{p_{i+1}} \frac{1}{\alpha_{i+1}} \\
&= L_{i+1}(t) \\
&= L(t, T_i, T_{i+1}).
\end{aligned}$$

This shows that over each elementary period $[T_i, T_{i+1}]$, the swap rate is just the LIBOR rate.

Now consider $[T_n, T_N]$ and let $w_i(t) = \frac{\alpha_i p_i(t)}{S_n^N(t)}$. Then,

$$\begin{aligned}
\sum_{i=n+1}^{N} w_i(t) L_i(t) &= \sum_{i=n+1}^{N} \frac{\alpha_i p_i(t)}{S_n^N(t)} L_i(t) \\
&= \sum_{i=n+1}^{N} \frac{\alpha_i p_i(t)}{S_n^N(t)} \frac{1}{\alpha_i} \left[\frac{p_{i-1}(t) - p_i(t)}{p_i(t)}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=n+1}^{N} \frac{p_{i-1}(t) - p_i(t)}{S_n^N(t)} \\
&= \frac{1}{S_n^N(t)}[p_n(t) - p_{n+1}(t) + p_{n+1} - p_{n+2} + \ldots + p_{n-1} - p_N] \\
&= \frac{p_n(t) - p_N(t)}{S_n^N(t)} \\
&= {R_n}^N(t).
\end{aligned}
$$

This shows that the swap rate is a stochastic combination of LIBOR rates.

However, if we model $L_i(t)$ as log-normal, it does not necessarily imply that ${R_n}^N(t)$ will be log-normal too, and vice versa. This shows that the LIBOR and swap market models are incompatible, implying that one cannot actually use Black type formuale to price both caps/floors and swaps simultaneously, in contrast with market practice. For at-the-money strike rates, the inconsistency is supposedly small. Swap market models are much more complex than LIBOR market models. A detailed account on how to recover swaption prices using the LIBOR rate model is given in Chapter 10 of [29].

Chapter 4

South African Market

4.1 Historical background

The discovery of the Witwatersrand goldfields in 1886 and the subsequent establishment of mining and financial companies triggered the need for a platform on which to trade shares. As a result, the Johannesburg Stock Exchange (JSE) was founded in November 1887 by Benjamin Woollan. From there on, the JSE went from strength to strength gaining membership of the Federation Internationale des Bourses de Valeurs (FIBV) and the African Stock Exchanges Association in 1963 and 1993, respectively. In April 1987, Rand Merchant Bank (RMB) started an informal futures market operating as both an exchange and a clearing house. In September of the following year, an agreement was reached to form SAFEX (South African Futures Exchange) and the SAFEX Clearing Company (Pty) Limited (Safcom). May 15 1996 saw the passing of the formal bond market from the JSE to the Bond Exchange of South Africa. This entity was licensed as a financial market in terms of the Financial Markets Control Act. Among others, most of the products listed on the JSE were futures contracts on the All Share, Gold and Industrial indices as well as the E168 Eskom bond. The introduction of options-on-futures in October 1992 triggered a huge market growth that saw volumes growing in the excess of 700% in 12 months and by December of 1993, volumes exceeded a record 1 million contracts. Currently options account for approximately 50% of volumes. May 2001 saw SAFEX and JSE Securities Exchange agreeing to a buy-out of SAFEX by the JSE with the JSE retaining the SAFEX branding and transferring the Financial Products business into an independent division known as SAFEX Financial Derivatives Division. After 119 years, the Johannesburg Securities Exchange is now a publicly traded company, with a listing on its own bourse. Attending the listing ceremony at the exchange's glass-clad offices in Sandton on 5 June 2006 was South Africa's deputy president, Phumzile Mlambo-Ngcuka. After long being a mutual institution owned by those who made use of it, the exchange was de-mutualized in July 2005, becoming an unlisted public company known as JSE Limited. Following just under a year of over-the-counter trade, the company is now listed and for the first time anybody who is not a stockbroker or an authorised user of the JSE can own shares. Public trading of JSE Limited shares commenced on the same listing day Monday (5 June 2006) morning at a price of R26 per share, raising some R2,1-billion.

4.1.1 The JIBAR rate

Each day at 10h30 each of the 14 South African and South African-based foreign banks are asked to provide the midpoint between Bid and Offer of their 1, 3, 6, 9 and 12 month deposit National (Negotiable) Certificate of Deposit (NCD) rates quoted as yield. In each category, e.g, in the 1 month category, the 14 rates are arranged in order. The top two and the bottom two are eliminated and the remaining 10 are averaged and rounded to 3 decimal places. The resulting rate is termed a k-month JIBAR rate where $k = 1, 3, 6, 9, 12$.

JIBAR stands for Johannesburg Inter Bank Agreed Rate. It is the rate at which banks buy and sell short-term money among themselves and is traditionally a wholesale and not a retail rate. The JIBAR is reset daily but for a swap contract, the 3-month JIBAR is reset every quarter and is fixed for the duration of the quarter. Let J_k represent the k-month SAFEX-JIBAR rate. Then

$$J_k = \frac{1}{n}\sum_{i=1}^{n} {Mpt_i}^k, k \text{ fixed},$$

where $k = 1, 3, 6, 9, 12,\ n = 10,\ Mpt_i = \frac{Bid_i + Offer_i}{2}$ is the midpoint corresponding to bank i.

4.2 Interest rate caps and floors: The South African context

In many circumstances, corporate treasurers in South Africa are hesitant to enter into interest rate derivative agreements which involve an element of optionality. The main deterrent factor is that many of them, besides the Black model, do not necessarily have other sophisticated pricing models to accurately price these derivatives. However, for many corporate treasurers, caps and floors have been the preferred method of achieving disaster insurance against incidents like the 1998 emerging markets crisis. This sterms from the fact that caps and floors are highly adaptable to the particular needs and requirements of companies wishing to manage and hedge against interest rate reset risk on interest-sensitive assets and liabilities. On the exercise date of the cap or floor agreement, the pre-specified strike rate is compared to the standard reference floating rate, that is the 3-month SAFEX-JIBAR rate. The interest differential is then applied to the contractually specified notional principal amount (amount to be borrowed/lent) in order to calculate the amount to be paid by the writer/seller to the holder/buyer (the settlement). The notional principal amount is normally at least R1 million.

Settlement of a single period cap/caplet is done in the following manner. The seller of a cap agrees to pay the buyer the difference between the fixed strike rate and the reference floating rate (JIBAR), based on the notional principal amount, when the JIBAR reset exceeds the fixed strike rate. Settlement occurs on each reset date according to the formula:

$$S = \frac{(J - K_c)Ld}{36500},$$

where S is the settlement amount in Rands, J is the JIBAR rate for that period/quarter, K_c is the cap strike rate, L is the notional principal amount, and d is the exposure period in days (usually 91 or 92).

In the majority of cases, settlement takes place in arrears, in which case the settlement amount is then present-valued to the exercise date. Consider the not-in-arrears case and for illustrative purposes, take a company that feels it might need to borrow R1 million in 3 months' time for a period of 3 months. However, this company fears that rates might go up and wishes to hedge against this risk. The company then buys a T3m-T6m at-the-money (ATM) caplet, i.e the right to borrow R1 million in 3 months' time for 3 months. Assume the following data:

Current 3-month SAFEX-JIBAR: 7.30%

T3m-T6m ATM caplet strike rate: 7.50%

3-month SAFEX-JIBAR in 3 months' time: 8.25%

Premium: R1 500

Then settlement amount is

$$S = \frac{(J - K_c)Ld}{36500}$$

$$= \frac{(8.25 - 7.50)1000000 \times 91}{36500}$$

$$= 1869.$$

The holder's (buyer) benefit is

S - Premium = R1869 - R1500 = R369.

A 3-month SAFEX-JIBAR of less than or equal to the strike would make the ATM caplet expire out-the-money and no settlement would take place. This would be the case if in the above we were considering an in advance caplet. By premium we mean the total cost to the client (corporate) of the full period of the cap. It is the sum of all the caplets, both in- and out-the-money making up the cap.

In a similar fashion, the settlement amount of a single period floor/floorlet is given by the formula:

$$S = \frac{(K_f - J)Pd}{36500}$$

where S is the settlement amount in Rands, J is the JIBAR rate for that period, K_f is the floor strike rate, L is the notional principal amount, and d is the exposure period in days.

In this case, the seller of a floor agrees to pay the buyer the difference between the fixed strike rate and the SAFEX-JIBAR, based on the notional principal amount, when the SAFEX-JIBAR rate resets below the fixed strike rate. Settlement also takes place on each reset date. To get a better feeling of this, take a company that expects a surplus cash receipt of R1 million in a month's time which it will wish to invest. The company fears rates will be lower in future and therefore decides to buy a T1m-T4m at-the-money floorlet with a maturity of 3 months, to hedge against the risk of losing money. For illustrative purposes, consider the following data:

Current 3-month SAFEX-JIBAR rate: 7.30%

T1m-T4m ATM floorlet strike rate: 7.35%

3-month SAFEX-JIBAR rate in 1 month's time: 6.95%

Premium: R2000

The settlement amount is therefore

$$\begin{aligned} S &= \frac{(K_f - J)Ld}{36500} \\ &= \frac{(7.35 - 6.95) \times 1000000 \times 91}{36500} \\ &= 997. \end{aligned}$$

The holder's benefit in this case is:

R2000 - R997 = R1003.

The need by most floating rate corporate borrowers to reset their debt quarterly or semi-annually leads them into wanting to fix borrowing rates for multiple periods. They would therefore prefer a string of caplets. A 1-year cap resetting against the 3-month SAFEX-JIBAR rate would therefore be a series of options on the 3×6, 6×9 and the 9×12 forward rate agreements (FRAs), all with a common strike.

Saying that a corporate treasurer purchases a 3-year cap resetting against 3-month SAFEX-JIBAR with a cap strike rate of $K_c\%$, means that for every 3-month reset period over the next 3 years, he will be reimbursed, by the seller, the differential recorded between the 3-month SAFEX-JIBAR rate and the cap rate of $K_c\%$, calculated on the notional principal amount. Settlement would only take place on those reset dates where the 3-month SAFEX-JIBAR exceeds the cap rate of $K_c\%$, otherwise the particular caplet would expire worthless.

4.2.1 Pricing caps, floors and collars

Each caplet/floorlet is priced from the implied 3-month forward rate for that period, from the yield curve. Hence, the at-the-money price of a caplet/floorlet is just the forward rate for that period. A strike price lower than that implied by the forward rate will result in an in-the-money caplet with both intrinsic and time values, whereas a strike price above the forward rate will result in an out-the-money caplet. Similarly as with most option-styled derivative instruments, the more time to expiry, the greater the time value inherent in the option. This means that a T3m-T6m period caplet has time value of 3 months while a T21m-T24m period caplet has time value of 21 months. Volatility (annualized) is another factor that affects the value of a cap/floor. There is a positive correlation between volatility and the price of both caps and floors. The more volatile the price or rate of an asset, the more likely it is to reach the option strike price, and so the more valuable the option. In brief, higher volatility implies higher option value. Standard option pricing theory postulates that the spot price or rate of the underlying follows a log-normal random walk. The fact that there are so many factors impacting on the price of a cap/floor makes it difficult for market-makers to hedge caps and floors. Basically, the pricing of caps and floors in the South African market follows an extension of the Black-Scholes option valuation formula and is done in the following manner.

All the major players in the South African cap/floor/swap market use the Black's formula and other models for the valuation of caps and floors. Next we recap on how they employ the Black formula. Suppose we have an interest rate cap with strike rate K and reset at times $t_1, t_2, \ldots, t_N$, with a final payment to be made at time t_{N+1}. If we let $\lambda_k = t_{k+1} - t_k$ and R be the λ_k maturity forward rate observed at time t_k, $1 \leq k \leq N$. Then the time-0 price of the kth caplet c_k is given by

$$c_k = \lambda_k L e^{-rt_{k+1}}[N(d_2)R - N(d_1)K], \tag{4.1}$$

where L is the nominal amount.
Similarly for a floor, the price of the kth floorlet f_k with strike K is given by

$$f_k = \lambda_k L e^{-rt(k+1)}[N(-d_2)K - RN(-d_1)]. \tag{4.2}$$

In both cases,

$$\begin{aligned} d_1 &= \frac{\ln \frac{R}{K} + \frac{\sigma^2}{2} t_k}{\sigma\sqrt{t_k}}, \\ d_2 &= d_1 - \sigma\sqrt{t_k}. \end{aligned}$$

r is the continuously compounded rate at the caplet/floorlet payment time t_{k+1}. The cap/floor price is the sum of the prices of the caplets/floorlets.

4.3 The SAFEX-JIBAR market model

Consider a fixed set of increasing maturities $T_0, T_1, \ldots, T_N$ such that $T_i - T_{i-1}$= exposure period in days. Define $\beta_i = \frac{T_i - T_{i-1}}{365}$, $i = 1, 2, \ldots, N$ as the day-count factor (usually 91/365 or 92/365). Denote by J_i the 3-month SAFEX-JIBAR rate corresponding to the period $[T_{i-1}, T_i]$. We can therefore define a caplet with strike K and resettlement dates $T_0, T_1, \ldots, T_N$ as a contract which at time T_i gives the holder a pay-off or settlement amount of

$$S_i = \beta_i \cdot \max[J_i - K, 0], \tag{4.3}$$

where J_i is the reference floating SAFEX-JIBAR rate for the period $[T_{i-1}, T_i]$; K is the caplet strike. β_i is normally termed the *tenor*. Both the floating and strike rates are in decimal form.

Thus, for a portfolio of N caplets we would have the following settlements:

$$
\begin{aligned}
S_1 &= \beta_1 \cdot \max[J_1 - K, 0] \\
S_2 &= \beta_2 \cdot \max[J_2 - K, 0] \\
S_3 &= \beta_3 \cdot \max[J_3 - K, 0] \\
\vdots &= \vdots \ddots \vdots \ddots \\
S_N &= \beta_N \cdot \max[J_N - K, 0]
\end{aligned}
$$

Since by definition, J_i is an average, for every $i = 1, 2, \ldots, N$, the JIBAR-SAFEX process J_i is a martingale under the corresponding forward measure Q^{T_i} on the interval $[T_{i-1}, T_i]$. (See Section 2.2.1). As mentioned earlier, standard option pricing theory postulates that the spot price or the rate of the underlying follows a log-normal random walk. If for each i the SAFEX-JIBAR rate $J_i(t)$ is log-normal under its measure, we assume $J_i(t)$ satisfies (2.20), then we have

$$
\begin{aligned}
\frac{dJ_i(t)}{J_i(t)} &= \sigma_i(t) dW^i(t) && (4.4) \\
J_i(T) &= J_i(t) e^{\int_t^T \sigma_i(s) dW^i(s) - \frac{1}{2}\int_t^T ||\sigma_i(s)||^2 ds}. \\
\Longrightarrow \ln\left(\frac{J_i(T)}{J_i(t)}\right) &= \int_t^T \sigma_i(s) dW^i(s) - \frac{1}{2}\int_t^T ||\sigma_i(s)||^2 ds. && (4.5)
\end{aligned}
$$

Define $q_i(t) = Le^{-r(T_i - t)}$ where r is the continuously compounded forward rate for the period $[T_{i-1}, T_i]$ and L is the notional amount. We extend the theory in Chapter to propose the following results the proofs of which follow without loss of generality from the proof of Proposition 2.8. These results should help practitioners in the South African market in the following manner:

1. The formulae for caps, floors and collars should help them price these instruments in a clearer way as they are purely JIBAR based.
2. The formulae for the Greeks should be of good help in hedging and risk management purposes.

Proposition 4.1 *In the SAFEX-JIBAR market, the time-t price of a caplet with strike K is given by*

$$\begin{aligned} Capl_i(t) &= \beta_i q_i(t) \left\{ J_i(t) N[d_1(t, T_{i-1})] - KN[d_2(t, T_{i-1})] \right\} \\ &= \beta_i q_i(t) \left\{ J_i(t) N[d_1] - KN[d_2] \right\} \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} d_1 &= \frac{1}{v_i(t,T)} \left\{ \ln\left(\frac{J_i(t)}{K}\right) + \frac{1}{2} v_i^2(t, T_{i-1}) \right\} \\ d_2 &= d_1 - v_i(t, T_{i-1}). \end{aligned}$$

Just as in Section 2.2.1,

$$m_i(t,T) = -\frac{1}{2} \int_t^T ||\sigma_i(s)||^2 ds$$

and

$$v_i^2(t,T) = \int_t^T ||\sigma_i(s)||^2 ds.$$

Definition 4.2 *A floorlet with strike K and resettlement dates $T_0, T_1, \ldots, T_N$ is a contract which at time T_i gives the holder a settlement amount of*

$$S_i = \beta_i \cdot \max[K - J_i(t), 0]. \tag{4.7}$$

Proposition 4.3 *In the SAFEX-JIBAR market, the price of a floorlet whose settlement amount is given by*

$$S_i = \beta_i \cdot \max[K - J_i(t), 0], \tag{4.8}$$

is given by the formula

$$Floorl_i(t) = \beta_i q_i(t)\{KN[-d_2] - J_i(t)N[-d_1]\} \tag{4.9}$$

where

$$\begin{aligned} d_1 &= \frac{1}{v_i(t,T)}\left\{\ln\left(\frac{J_i(t)}{K}\right) + \frac{1}{2}v_i^2(t,T_{i-1})\right\} \\ d_2 &= d_1 - v_i(t,T_{i-1}). \end{aligned}$$

where σ_i is the volatility of the interest rate of the period (t_{i-1}, t_i).

Proposition 4.4 *The time-t price of a SAFEX-JIBAR collar with resettlement dates $T_0, T_1, \ldots, T_N$ is given by*

$$Collar_i(t) = \beta_i q_i(t)\left\{[J_i(t)N(d_1^c) - K_c N(d_2^c)] - [K_f N(-d_2^f) - J_i(t)N(-d_1^f)]\right\},$$

where K_c and K_f are the cap and floor strike rates respectively,

$$\begin{aligned}
d_1^c &= \frac{1}{v_i(t,T)}\left\{\ln\left(\frac{J_i(t)}{K_c}\right)+\frac{1}{2}v_i^2(t,T_{i-1})\right\} \\
d_2^c &= d_1 - v_i(t,T_{i-1}), \\
d_1^f &= \frac{1}{v_i(t,T)}\left\{\ln\left(\frac{J_i(t)}{K_f}\right)+\frac{1}{2}v_i^2(t,T_{i-1})\right\} \\
d_2^f &= d_1 - v_i(t,T_{i-1}),
\end{aligned}$$

σ_i *is the volatility of the interest rate of the period* (t_{i-1}, t_i).

Equations (4.6) and (4.9) show that the numeraire for the pricing of caps and floors in the JIBAR market is $\beta_i q_i(t)$.

4.3.1 The Greeks

In this section, we intend to derive formulae for some hedging measures for our model. Most traders employ sophisticated hedging schemes which involve the calculation of such measures as delta, gamma and vega. The delta of an option measures the rate at which the option price changes with respect to the price of the underlying forward rate. Gamma is the rate of change of the option's delta with respect to the forward rate. Vega is the rate of change of option price with respect to the volatility of the underlying. If vega is high in absolute terms, then the option value is sensitive to small changes in volatility. In contrast, if vega is small in absolute terms, volatility changes have relatively little impact on the value of the option. We will recall that[17]

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

and that since

$$\begin{aligned}
\ln \frac{J_i N'(d_1)}{K_c N'(d_2)} &= \ln \frac{J_i}{K_c} + \frac{1}{2}[d_1^2 + u_i^2 - 2v_i d_1 - d_1^2] \\
&= \ln \frac{J_i}{K_c} + \frac{1}{2}[v_i^2 - 2\ln \frac{J_i}{K_c} - v_i^2] \\
&= 0
\end{aligned}$$

we have

$$J_i N'(d_1) - K_c N'(d_2) = 0.$$

This fact will help us deduce our measures in the following manner. For a caplet,

$$\begin{aligned}
\Delta &= \frac{\partial C}{\partial J_i} \\
&= \beta_i q_i(t)\{N(d_1) + J_i N'(d_1) \cdot \frac{\partial d_1}{\partial J} - K_c N'(d_2) \cdot \frac{\partial d_2}{\partial J_i}\} \\
&= \beta_i q_i(t)\{N(d_1) + J_i N'(d_1) \cdot \frac{1}{J_i v_i} - K_c N'(d_2) \cdot \frac{1}{J_i v_i}\} \\
&= \beta_i q_i(t)\{N(d_1) + \frac{J_i N'(d_1) - K_c N'(d_2)}{J_i v_i}\} \\
&= \beta_i q_i(t) N(d_1).
\end{aligned}$$

$$
\begin{aligned}
\Gamma &= \frac{\partial^2 C}{\partial {J_i}^2} \\
&= \beta_i q_i(t) N'(d_1) \cdot \frac{\partial d_1}{\partial J_i} \\
&= \frac{\beta_i q_i(t) N'(d_1)}{J_i v_i}. \\
\text{vega} &= \frac{\partial C}{\partial v_i} \\
&= \beta_i q_i(t) \{ J_i N'(d_1) \cdot \frac{\partial d_1}{\partial v_i} - K_c N'(d_2) \cdot \frac{\partial d_2}{\partial v_i} \} \\
&= \beta_i q_i(t) \{ J_i N'(d_1)(-\frac{1}{{v_i}^2} \ln \frac{J_i}{K_c} + v_i) - K_c N'(d_2)(-\frac{1}{{v_i}^2} \ln \frac{J_i}{K_c} + v_i - 1) \} \\
&= \beta_i q_i(t) \{ J_i N'(d_1)(-\frac{1}{{v_i}^2} \ln \frac{J_i}{K_c} v_i) - K_c N'(d_2)(-\frac{1}{{v_i}^2} \ln \frac{J_i}{K_c} + v_i) + K N'(d_2) \} \\
&= \beta_i q_i(t) \{ (-\frac{1}{{v_i}^2} \ln \frac{J_i}{K_c} + v_i)[J_i N'(d_1) - K_c N'(d_2)] + K N'(d_2) \} \\
&= \beta_i q_i(t) K N'(d_2).
\end{aligned}
$$

Similarly, it can be shown that for floorlets,

$$
\begin{aligned}
\Delta &= -\beta_i q_i(t) N(-d_1). \\
\Gamma &= \frac{\beta_i q_i(t) N'(-d_1)}{J_i v_i}. \\
\text{vega} &= \beta_i q_i(t) K N'(-d_2).
\end{aligned}
$$

Note that the delta, gamma and vega of a cap/floor is simply the arithmetic sum of the respective delta, gamma and vega for the caplets involved.

Chapter 5

Computational Analytics

In this Chapter, we intend to perform some numerical comparisons between the JIBAR model and some well known models for pricing caps and floors.

The data used in the following tests was obtained from Rand Merchant Bank. Historical data on JIBAR rates was obtained from the SAFEX website. The following MATLAB codes were used to calculate the price of the cap/floor with the following specifications:

Instrument: Quarterly resetting year-long cap/floor

Notional amount: 100 000 000

Cap/Floor strike rate: 12.95%

Volatility: 15%

The inputs to the JIBAR code are defined below:

1. J: the 3 month JIBAR

2. K_f is the floorlet strike

3. K_c is the caplet strike

4. v is the volatility (flat)

5. L is the notional amount in South African Rands

6. "Days" is the number of days (91 or 92 for each quarterly resetting cap/floor)

7. β is the tenor

8. D_f is the discount factor

9. r is the continuously compounded forward rate.

$J =; K_c =; v = 0.15; L = 100\ 000\ 000; Days =; \beta = Days/365;$

$t = 0.25; r =; q_i = L * D_f;$

JIBAR Caplet

$d_1 = 1/(v^2 * \beta) * (\log(J/K_c) + 0.5 * \beta * v^2);$

$d_2 = d_1 - (v^2 * \beta);$

$N_1 = 0.5 * (1 + erf(d_1/(\sqrt{(2)})));$

$N_2 = 0.5 * (1 + erf(d_2/(\sqrt{(2)})));$

Caplet value $= \beta * q_i * (J * N_1 - K_c * N_2);$

$disp($'Caplet value is'$), disp($Caplet value$)$

JIBAR Floorlet

$d_1 = 1/(v^2 * \beta) * (\log(J/K_f) + 0.5 * v^2);$

$d_2 = d_1 - (v^2 * \beta);$

$N_1 = 0.5 * (1 + erf(-d_1/(\sqrt{(2)})));$

$N_2 = 0.5 * (1 + erf(-d_2/(\sqrt{(2)})));$

$\text{Floorletvalue} = \beta * q_i * (K_f * N_2 - J * N_1);$

$disp('\text{Floorlet value is}'), disp(\text{Floorlet value})$

The Black formula was coded as follows:

1. $L = 100\ 000\ 000$; Notional amount;

2. $K =$; The cap strike;

3. $R = 0.07$; the zero curve is flat at this rate/floating rate

4. r=; the continuously compounded zero rate (for all maturities)

5. T=; starting in T years

6. $\lambda = 0.25$; =tenor for quarterly resetting caplet

7. σ=; volatility

Black Caplet

$d_1 = (\log(R/K) + (0.5 * \sigma^2) * T)/(\sigma * \sqrt{T});$

$d_2 = d_1 - \sigma * \sqrt{T};$

$N_1 = 0.5 * (1 + erf(d_1/\sqrt{2}));$

$N_2 = 0.5 * (1 + erf(d_2/\sqrt{2}));$

$\text{Capletvalue} = \lambda * L * \exp(-r * t) * (R * N_1 - K * N_2)$

Black Floorlet

$d_1 = (\log(R/K) + (0.5 * \sigma^2) * T)/(\sigma * \sqrt{T});$

$d_2 = d_1 - \sigma * \sqrt{T};$

$N_1 = 0.5 * (1 + erf(-d_1/\sqrt{2});$

$N_2 = 0.5 * (1 + erf(-d_2/\sqrt{2}));$

$\text{Floorletvalue} = \lambda * L * \exp(-r * t) * (K * N_2 - R * N_1)$

The expression $\exp(-r * t)$ is the discount factor.

In Fig.5.1, the 3-month JIBAR rates are those calculated on the start date and effective for the next 3 months. It is important to note that the forward rates are increasing with time. This is not the case with the JIBAR rates. This fact will be reflected in the corresponding prices as reflected in the upcoming Figures. Settlement for the in advance cap/floor is made at the start date.

Caplet/Floorlet	Start date	Maturity date	Days	3-month JIBAR	Forward rate
1	16 Feb 04	17 May 04	91	7.750	7.949
2	17 May 04	16 Aug 04	91	7.669	8.016
3	16 Aug 04	15 Nov 04	91	7.243	8.290
4	15 Nov 04	15 Feb 05	92	7.450	8.770

Figure 5.1: Data on an RMB year-long in advance cap/floor starting on 16 Feb 2004 and ending on 15 Feb 2005. The JIBAR rates were obtained from the SAFEX historical data file. Despite concerted efforts to find out the reason for the increasing nature of the forward rate process or how it was obtained, *bank confidentiality issues* were repeatedly cited and no further explanation was given. As mentioned above, this process will impact on all future figures where it is inherent.

Caplet	Disc. factor	JIBAR price	RMB	DerivaGem
1	0.9781	11.9070	0.00	
2	0.9589	21.9339	0.00	
3	0.9395	4.3927	1.34	
4	0.9192	9.8527	154.86	
Cap price		48.0863	156.20	0.00

Figure 5.2: Comparison of the price of the in advance cap using different models. Obviously this cap will not be exercised. It is important to note that the price of the last caplet looks incorrect. Again, the data has been accepted and presented in good faith.

Floorlet	Disc. factor	Black price	JIBAR price	RMB	DerivaGem	Average
1	0.9781	1 222 925	1 268 075	1 219 536		
2	0.9589	1 182 871	1 263 625	1 179 512		
3	0.9392	1 094 670	1 336 763	1 091 132		
4	0.9192	961 103	1 260 446	968 020		
Floor price		4 461 570	5 128 909	4 458 200	5 469 006	4 879 421

Figure 5.3: Comparison of the price of the in advance floor using different models. This floor is exercisable.

Caplet/Floorlet	Start date	Maturity date	Days	3-month JIBAR	Forward rate
1	16 Feb 04	17 May 04	91	8.097	7.949
2	17 May 04	16 Aug 04	91	7.389	8.016
3	16 Aug 04	15 Nov 04	91	7.450	8.290
4	15 Nov 04	15 Feb 05	92	7.300	8.770

Figure 5.4: Data on an RMB year-long in-arrears cap/floor starting on 16 Feb 2004 and ending on 15 Feb 2005. The JIBAR rates were obtained from the SAFEX historical data file.

Caplet	Disc. factor	JIBAR price	RMB	DerivaGem
1	0.9781	0.00	0.00	
2	0.9589	0.00	0.00	
3	0.9395	0.00	1.34	
4	0.9192	0.00	154.86	
Cap price		0.00	156.20	0.00

Figure 5.5: Comparison of the price of the in-arrears cap using different models. This cap will not be exercised. The reason for the discrepancy in the price of the last caplet follows from Figure 5.2.

Floorlet	Disc. factor	Black price	JIBAR price	RMB	DerivaGem	Average
1	0.9781	1 222 925	1 183 429	1 219 536		
2	0.9589	1 182 871	1 329 458	1 179 512		
3	0.9392	1 094 670	1 287 862	1 091 132		
4	0.9192	961 103	1 309 042	968 020		
Floor price		4 461 570	5 109 790	4 458 200	5 469 006	4 874 642

Figure 5.6: Comparison of the price of the in-arrears floor using different models. This floor will be exercisable.

In Figure 5.2, the Black price is the price as calculated by a MATLAB code of the Black formula. DerivaGem[19] is a software that can be used to price interest rate instruments, including caps and floors using the Black model for a European-type option. The only inputs to this software are (for a cap/floor): settlement frequency, principal, start and end dates in years, strike, pricing model, volatility. The RMB price was reportedly also calculated based on the Black model but clearly their caplet prices point to some error. The software used is unknown. Note that (Fig. 5.2) DerivaGem gives a cap price consistent with economic reality (for a cap strike as high as 12.95%) while the JIBAR price is not far off the mark.

The increasing nature of the RMB prices of the caplets seem to emanate from the increasing forward rate process and the decreasing discount process. The discount factors were also obtained from RMB. It is important to note that in Figure 5.3, the Black and the RMB prices are comparable. Again the decreasing nature of the floorlet prices can be attributed to the same fact as in Figure 5.2. The 3-month JIBAR rates in Figure 5.4 are of the next period. For example, 8.097% is the JIBAR rate calculated on 17 May 2004 and valid for the next three months. Settlement is made on maturity date, i.e on 17 May 2004 for caplet 1, for example. However, the forward rates remain unchanged. Since it is not known whether the RMB prices are for an in advance or in arrear cap/floor, Figures 5.5 and 5.6 contain these as they were in Figures 5.2 and 5.3. It is not a simple task to get all the information one needs from the local banks as each time *confidentiality issues* are raised when further enquiries are made. DerivaGem also does not distinguish between the in advance-in arrear cases. So the JIBAR prices are the only ones reflecting this fact. Again, it is important to note that DerivaGem and JIBAR methods give a cap price consistent with economic sense for the given data. Obviously the seller of a floor wants a bigger price and the two methods provide exactly that. Moreso, the corresponding floor prices for these two methods are relatively comparable, a fact which is also notable in the RMB and Black prices. See Figure 5.6.

Why are the prices different? This can be attributed to a variety of reasons, including but not limited to model risk and minor differences in inputs. The major question then arises: What/which is the best price? Obviously the best price depends on your position. The seller of a floor wants a higher price while the buyer of a cap is interested in paying less. Instead of sticking to one method, it would be advisable to combine methods and maybe the average of the resulting prices can be deemed *best price*. The last columns in Figures 5.3, 5.5 and 5.6 give the average prices.

5.0.2 Concluding Remarks

The LIBOR model uses the LIBOR. The analytic results in this Chapter seem to support our earlier claim in Section 4.3. The JIBAR model can provide South African practitioners with a much more relevant and alternative model to price caps, floors and collars in a South African context. Firstly, the formulae for caps, floors and collars should help practitioners to price these instruments in a clearer way as they are purely JIBAR based. Secondly, the JIBAR based Greek formulae deduced can be of substantial help for hedging and risk management purposes.

5.1 SAFEX-JIBAR Swap and Swaptions models

Consider an elementary period $[T_i, T_{i+1}]$, that is, a quarter of a year. Then by Section 3.4, the swap rate $R_i^{i+1}(t)$ is just the JIBAR rate $J_t(t) = J_i^{i+1}(t)$.

Recall: $S_i^{i+1}(t)$ is the accrual factor or the present value at time t of a self-financing portfolio of bonds.

Similarly as in Chapter 3,

$$J_i^{i+1}(T_i) = J_i^{i+1}(t)e^{\int_t^{T_i} \sigma_{i,i+1}(t)dW_i^k(s) - \frac{1}{2}\int_t^{T_i} ||\sigma_{i,i+1}(s)||^2 ds}. \tag{5.1}$$

And by the deterministic nature of $\sigma_{i,i+1}$, conditional on $\mathcal{F}_t$, $J_i^{i+1}(T_i)$ is log-normal,

$$J_i^{i+1}(T_i) = J_i^{i+1}(t)e^{Y_i^{i+1}(t,T_i)}, \tag{5.2}$$

where $Y_i^{i+1}(t, T_i)$ is normally distributed with expected value

$$m_i^{i+1}(t, T_i) = -\frac{1}{2}\int_t^{T_i} ||\sigma_{i,i+1}(s)||^2 ds, \tag{5.3}$$

and variance

$$v_i^{i+1^2}(t, T_i) = \int_t^{T_i} ||\sigma_{i,i+1}(s)||^2 ds. \tag{5.4}$$

Proposition 5.1 *The arbitrage-free SAFEX-JIBAR price of a payer swap with swap rate K is*

$$PS_i^{i+1}(t, K) = \left[J_i^{i+1}(t) - K\right] S_i^{i+1}(t). \tag{5.5}$$

Similarly, the price of a receiver swap with strike K is given by

$RS_i^{i+1}(t) = \left[K - J_i^{i+1}(t)\right] S_i^{i+1}(t).$

The SAFEX-JIBAR swap market model price of a $T_i \times (T_{i+1} - T_i)$ swaption is

$$PSN_i^{i+1}(t) = S_i^{i+1}(t)E^{i,i+1} \max\left[\left[J_i^{i+1}(T_i) - K, 0\right] |\mathcal{F}_t\right], 0 \leq t \leq T_i. \tag{5.6}$$

Proposition 5.2 *In the swap market model, the* $T_i \times (T_{i+1} - T_i)$ *payer swaption price with strike* K *is given by*

$$PSN_i^{i+1}(t) = S_i^{i+1}(t)\left\{J_i^{i+1}(t)N[d_1] - KN[d_2]\right\}, \tag{5.7}$$

where

$$\begin{aligned} d_1 &= \frac{1}{v_i^{i+1}\sqrt{T_i - t}}\left[\ln\left(\frac{J_i^{i+1}(t)}{K}\right) + \frac{1}{2}{v_i^{i+1}}^2\right] \\ d_2 &= d_1 - \sigma_{i,i+1} \end{aligned}$$

which is of the Black-76 type.

Similarly, the price of a receiver swaption with strike K is given by

$$RSN_i^{i+1}(t) = S_i^{i+1}(t)\left[KN(-d_2) - J_i^{i+1}(t)N(-d_1)\right]. \tag{5.8}$$

The proofs of the above propositions follow from the proof of Proposition 3.10.

5.2 Calibration and Other Issues

Generally, there is a one-to-one correspondence between option prices and the volatility parameter. So far we have considered constant volatility. The process of obtaining the appropriate volatility parameter for pricing an instrument is called *calibration*. Given some data, say market data, and a model to calibrate, one seeks for the most appropriate volatility such that the model produces the market prices.

5.2.1 The Hull-White and Ho-Lee Models

An often used model in interest rate modeling is the Hull-White model. Two main reasons justify its popularity. Firstly, it provides closed-form solutions for bond and plain vanilla European option pricing and hence there is no need for time-consuming simulations. Secondly, and more importantly, this model, in contrast to equilibrium models such as the Vasicek, Cox-Ross-Ingersoll models, belongs to the class of no-arbitrage interest rate models.This means that it succeeds in fitting a given term-structure by having at least one time dependent parameter. In this way, today's bond prices can be perfectly matched.

Like any other model, the Hull-White has its own problem in that it sometimes results in a negative interest rate. It has however been shown that with up-to-date calibrated parameters which are used for a shorter period, the probability of obtaining negative interest rates is minimized.

Hull and White (1990) showed that the instantaneous interest rate follows a mean-reverting process also known as an Ornstein-Uhlenbeck process:

$$dr(t) = [\theta(t) - a(t)r(t)]dt + \sigma(t)dz(t) \tag{5.9}$$

or

$$dr(t) = a(t)\left(\frac{\theta(t)}{a(t)} - r(t)\right)dt + \sigma(t)dz(t) \tag{5.10}$$

where $z(t)$ is a standard Brownian motion under the risk-neutral measure Q, and, $a(t), \sigma(t)$ and $\theta(t)$ are time dependent parameters. $a(t)$ is the rate of mean reversion where the mean is $\theta(t)/a(t)$, and $\sigma(t)$ is the volatility. The function $\theta(t)$ can be calculated from the initial term structure according to the formula

$$\theta(t) = F_t(0,t) + aF(0,t) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right), \tag{5.11}$$

where $F(0,t)$ is the instantaneous forward rate curve observed in the market at time zero with maturity t, and F_t is the first derivative with respect to time. Here $a(t) = a$ and $\sigma(t) = \sigma$ and the t has been dropped for abbreviation purposes.

Bond prices at time t in the Hull-White model are given by

$$P(t,T) = \hat{A}(t,T)e^{-\hat{B}(t,T)R(t)} \tag{5.12}$$

where

$$\begin{aligned}\hat{A}(t,T) &= \exp\left\{\ln\frac{P(0,T)}{P(0,t)} - \frac{B(t,T)}{B(t,t+\delta t)}\ln\frac{P(0,t+\delta t)}{P(0,t)}\right\} \times \\ &\times \exp\left\{-\frac{\sigma^2}{4a}(1-e^{-2at})B(t,T)[B(t,T)-B(t,t+\delta t)]\right\}\end{aligned}$$

and $R(t)$ is the δt-period rate at time t, and

$$\hat{B}(t,T) = \frac{B(t,T)}{B(t,t+\delta t)}\delta t, \tag{5.13}$$

$$B(t,T) = \frac{1-e^{-a(T-t)}}{a}, \tag{5.14}$$

$$P(0,T) = e^{-R(0)T}, \tag{5.15}$$

$$P(0,t) = e^{-R(0)t}. \tag{5.16}$$

$P(0,T)$ and $P(0,t)$ can be observed in the market.

If in the Hull-White $a(t) = 0$, we get the Ho-Lee model. In this case, the expression for the price of a zero-coupon bond at time t in terms of the δt-period interest rate $R(t)$ is

$$P(t,T) = \hat{A}(t,T)e^{-R(t)(T-t)} \tag{5.17}$$

where

$$\hat{A}(t,T) = \exp\left\{\ln\frac{P(0,T)}{P(0,t)} - \frac{T-t}{\delta t}\ln\frac{P(0,t+\delta t)}{P(0,t)} - \frac{1}{2}\sigma^2 t(T-t)[(T-t)-\delta t]\right\}.$$

In order to use the Hull-White model or the Ho-Lee model, we need to find credible parameter values for a and σ. The process of obtaining these values is called *calibration*. In our case, we intend to use the Ho-Lee so the parameter of interest is σ.

5.2.2 The Standard Market Model

Consider a cap/floor expiring at time T, with principal L and cap/floor rate R_K. Here, the subscript K only serves to relates to the strike. Define R_k as the interest rate for the period $[t_k, t_{k+1}]$ observed at time t_k, $1 \leq k \leq n$. Here t_i, $1 \leq i \leq n$ are the reset dates with $t_{n+1} = T$ and $\delta_k = t_{k+1} - t_k$.

In [19], Hull showed that a standard market model for a cap/floor is given by

$$\text{caplet price} = L\delta_k P(0,t_{k+1})[F_k N(d_1) - R_K N(d_2)] \quad (5.18)$$

$$\text{floorlet price} = L\delta_k P(0,t_{k+1})[R_K N(-d_2) - F_k N(-d_1)] \quad (5.19)$$

where

$$d_1 = \frac{\ln(F_k/R_K) + {\sigma_k}^2 t_k/2}{\sigma_k\sqrt{t_k}}$$

$$d_2 \quad = \quad d_1 - \sigma_k\sqrt{t_k},$$

and F_k is the forward rate for the period $[t_k, t_{k+1}]$, σ_k is the volatility of R_k. $P(0, t_{k+1})$ is the price at time 0 of a zero coupon bond maturing at time t_{k+1}.

The put-call parity relationship

$$\text{cap price} = \text{floor price} + \text{swap value} \tag{5.20}$$

holds and will help us in determining the swap values from cap and floor prices.

The time t_k value of a zero-coupon bond that pays $L(1 + R_K\delta_k)$ at time t_{k+1} is

$$P(t_k, t_{k+1}) = \frac{L(1 + R_K\delta_k)}{1 + \delta_k R_k}. \tag{5.21}$$

Now, consider a swap option that lasts n years starting in T' years. The cashflows are received m times per year. The payment dates are $T_1, T_2, \ldots, T_{mn}$. If s_0 is the forward swap rate and s_K is the strike rate and σ is the volatility, then, defining A as the value of a contract that pays $1/m$ at times $T_i, 1 \leq i \leq mn$, the value of the swaption is given by

$$\text{payer swaption value} = LA[s_K N(-d_2) - s_0 N(-d_1)] \tag{5.22}$$

where L is the principal, and

$$d_1 = \frac{\ln(s_0/s_K) + \sigma^2 T'/2}{\sigma\sqrt{T'}} \tag{5.23}$$

$$d_2 = d_1 - \sigma\sqrt{T'} \tag{5.24}$$

$$A = \frac{1}{m}\sum_{i=1}^{mn} P(0, T_i). \tag{5.25}$$

For the receiver swaption,

$$\text{receiver swaption value} = LA[s_0 N(d_1) - s_K N(d_2)]. \tag{5.26}$$

5.2.3 Calibration Procedure

Among others, in this section we generate bond and caplet prices using Hull's standard market model and calibrate the LIBOR model to the cap curve, i.e determine the *implied volatilities* σ_i's which can then be used to assess the volatility most appropriate for pricing the instrument under consideration. Having done that, we calibrate the Ho-Lee model to the bond curve obtained by our standard market model. We numerically compute caplet prices using the Black-76 formula for caplets seen in Chapter 2 and compare these prices to the ones obtained using the standard market model. Finally we compute and compare swaption prices obtained by our standard market model and by the LIBOR model.

Consider a contract that caps/floors the interest on a principal loan amount of $L = 10000$ at $R_K = 8\%$ per annum starting now. Consider $\sigma_k = 0.20$, $\delta_k = 0.25$ and $T = 5$. Suppose the continuously compounded rates F_k are given in Fig. 5.7. The interest rates for the δt period are given in column 2 of Figure 5.8.

On the other hand, for our swaption numerics, we will consider a 5 year swaption starting in one year, i.e $T' = 1$. Thus $m = 2$, $n = 5$ and $mn = 10$. The payments are semiannually. The other parameters are as in the cap/floor case.

In Fig 5.13, the LIBOR model is calibrated to the cap curve as given by column 4 of Fig 5.8. That is, the caplet prices are equated to the LIBOR model keeping all the other parameters constant except the the volatility. The resulting "appropriate" volatilities are given here in decimal form. The "NaN" results from division by 0.

Column 2 of Fig 5.14 is generated from Equation (5.21). With all the other parameters fixed, the Ho-Lee model, Equation (5.17), is equated to the second column of Fig 5.14. The resulting bond volatilities are given above in the last column. In Fig 5.16, the forward rates are given. Column 3 is generated from Equation (5.22).

Column 4 is generated by the LIBOR model seen Chapter 2.

5.2.4 Concluding Remarks

Firstly, it is evident that the caplet price curves in Figs. 5.8 and 5.9 manifest the same behaviour. The results of Fig. 5.12 point to a decrease in volatilities with an increase in maturities. Figs. 5.13 and 5.14 suggest that as the bond prices steadily increase, the bond price volatilities curve seem to follow an upward opening parabola. More so, medium-term bonds seem to have lower volatilities than shorter and longer ones. Figs 5.16 and 5.17 suggest consistence between the two models under consideration there. One is just a slight vertical shift of the other. The difference might be attributed to the computational effects filtered in by the forward swap rates (See Chapter 3).

The interest rates for each period were randomly generated using MATLAB's rand function. The bond prices $P(0, t_{k+1})$ are generated according to Equation (5.16). The caplet, floorlet and swap prices are generated according to Equations (5.18), (5.19) and (5.20) respectively. The last column is a result of the Black's formula for caplets seen in Chapter 2.

Continuously compounded rates F_k
0.07
0.075
0.08
0.085
0.09
0.095
0.1
0.105
0.11
0.115
0.12
0.125
0.13
0.135
0.14
0.145
0.15
0.155
0.16
0.165.

Figure 5.7: The above were generated by a MATLAB code starting from 0.07 and ending at 0.165 with an increment of 0.005.

Maturities	Int Rates R_k	$P(0, t_{k+1})$	Caplet Prices	Floorlet Prices	Swap value	Black Caplet
0	0.0894	0.9778	0	24.4473	-24.4473	76.47
0.25	0.0896	0.9562	2.8971	14.8495	-11.9524	76.09
0.5	0.0897	0.9350	10.5411	10.5411	0	77.89
0.75	0.0896	0.9143	19.5150	8.0864	11.4286	82.77
1.0	0.0893	0.8943	28.8575	6.4988	22.3587	88.27
1.25	0.0889	0.8752	38.2161	5.3976	32.8185	93.81
1.5	0.0883	0.8568	47.4382	4.5978	42.8404	98.90
1.75	0.0876	0.8393	56.4501	3.9966	52.4535	103.60
2.0	0.0868	0.8225	65.2212	3.5327	61.6885	107.84
2.25	0.0860	0.8065	73.7355	3.1667	70.5688	111.74
2.5	0.0852	0.7912	81.9939	2.8728	79.1211	115.24
2.75	0.0843	0.7765	89.9917	2.6329	87.3588	118.51
3.0	0.0835	0.7624	97.7321	2.4346	95.2975	121.54
3.25	0.0827	0.7486	105.2076	2.2684	102.9392	124.45
3.5	0.0820	0.7353	112.4241	2.1276	110.2966	127.19
3.75	0.0813	0.7223	119.3755	2.0071	117.3684	129.80
4.0	0.0808	0.7095	126.0612	1.9031	124.1581	132.27
4.25	0.0803	0.6969	132.4738	1.8124	130.6613	134.62
4.5	0.0798	0.6844	138.6223	1.7329	136.8893	136.80
4.75	0.0795	0.6721	144.4907	1.6625	142.8281	138.84
5.0						

Figure 5.8: Data generated from the standard market model. R_k is the interest rate for the period $[t_k, t_{k+1}]$. $P(0, t_{k+1})$ is the time-0 price of a zero-coupon bond maturing at time t_{k+1}. The caplet, floorlet and swap values are calculated by the standard market model.

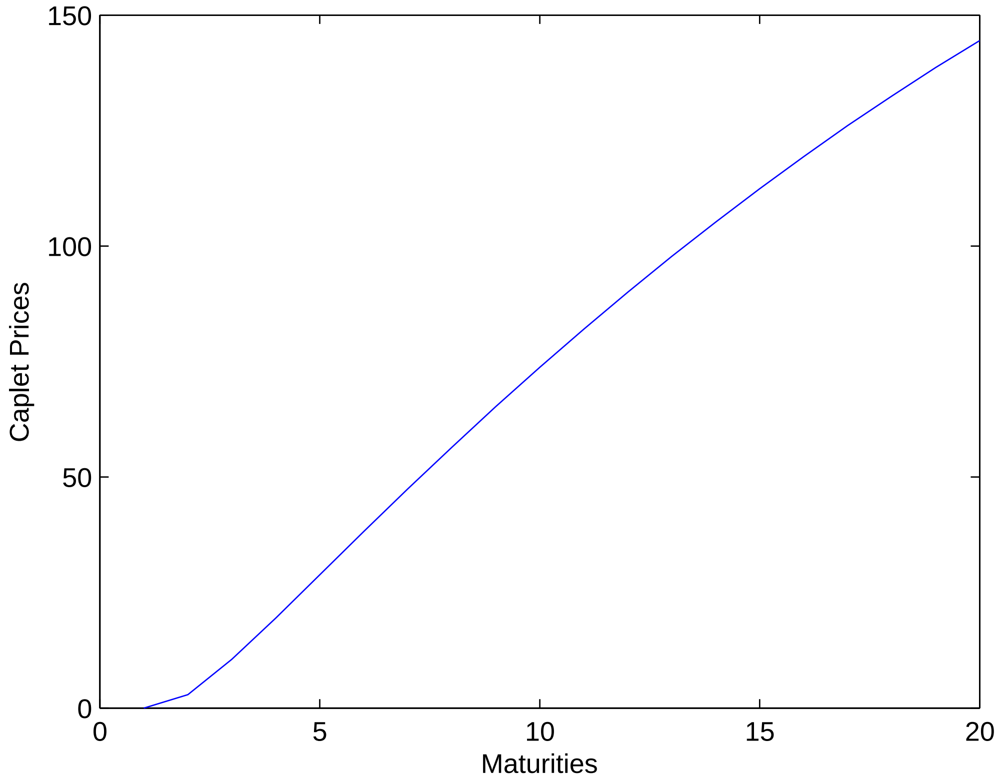

Figure 5.9: Caplet prices curve as generated by the standard market model. This is column 1 against column 4 in Fig 5.7.

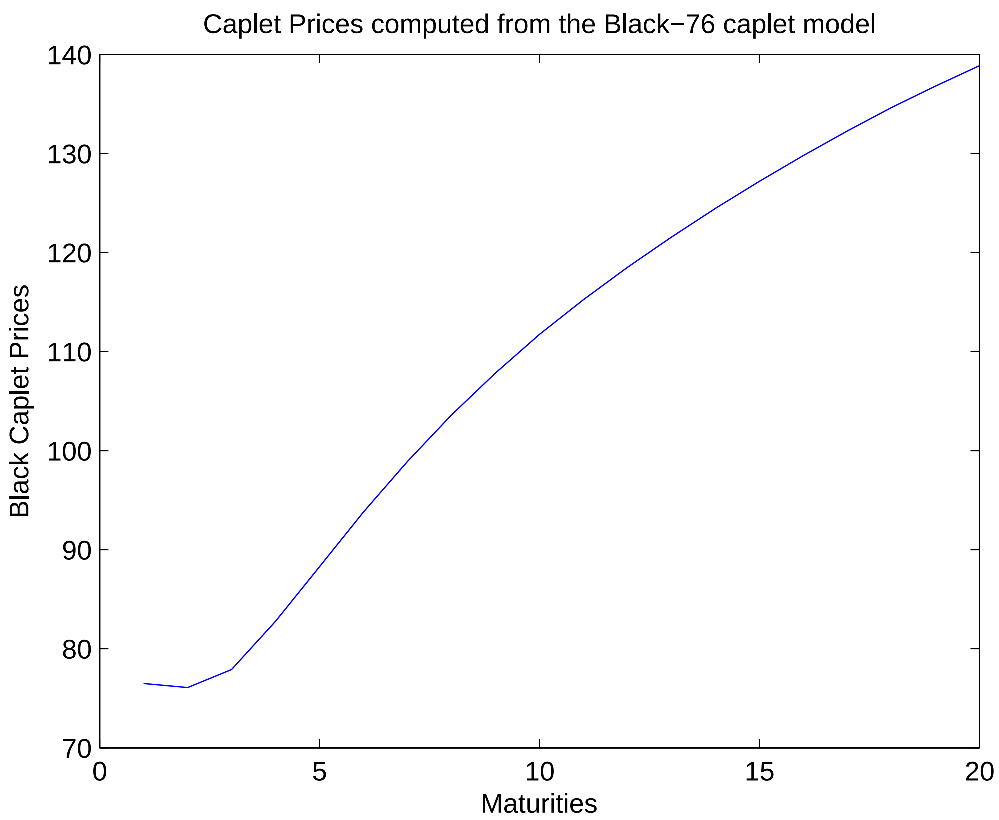

Figure 5.10: Caplet prices curve as generated by the Black-76 model. This is column 1 against the last column of Fig 5.8.

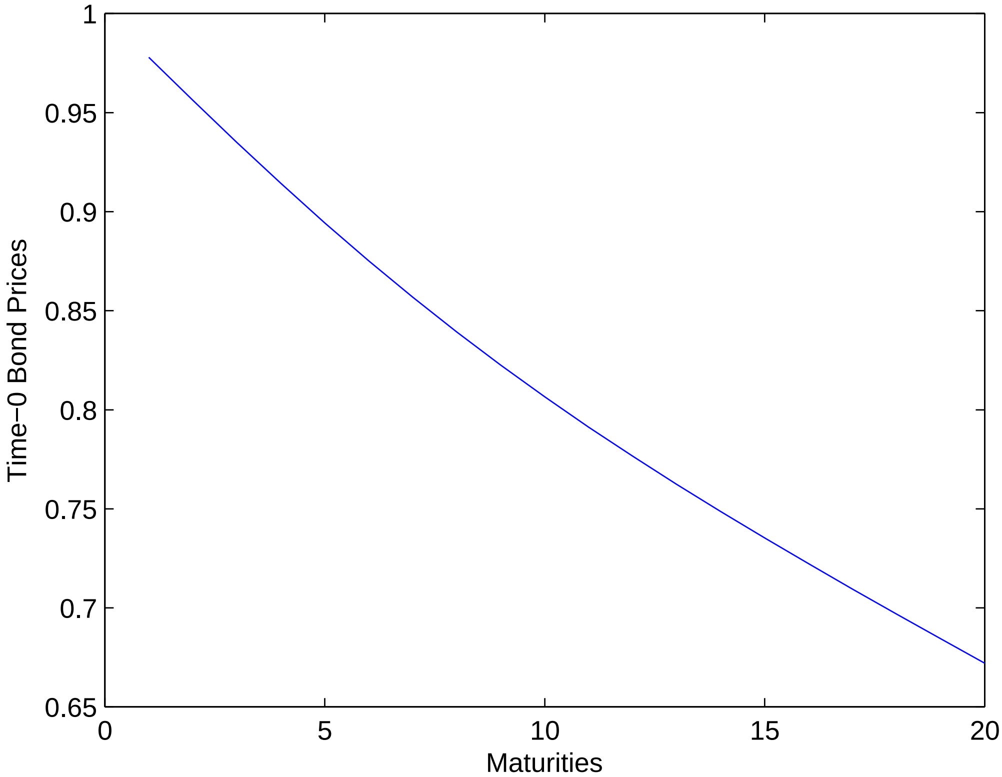

Figure 5.11: Time-0 zero-coupon bond curve. This results from plotting Column 1 against column 3 as seen in Fig 5.8.

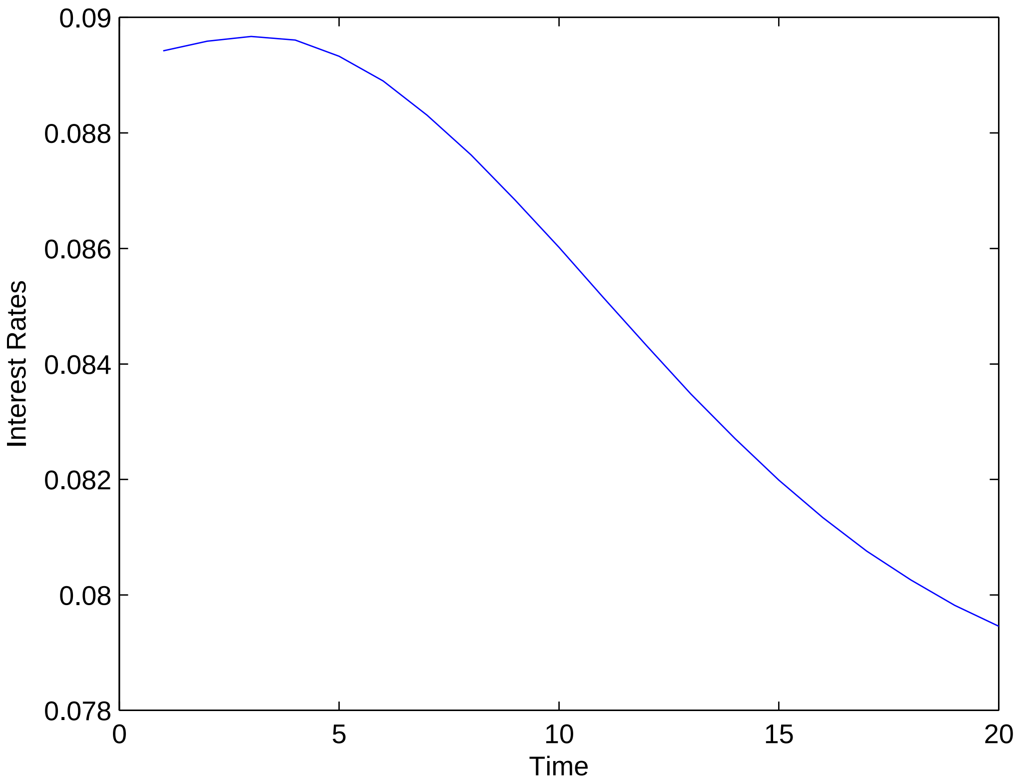

Figure 5.12: Term structure of interest rates as seen in column 2 of Fig 5.8.

Caplet Maturities	Caplet Volatilities as decimals
0	NaN
0.25	0.5612
0.50	0.5000
0.75	0.4338
1.00	0.3887
1.25	0.3191
1.50	0.1865
1.75	0.1253
2.00	0.0896
2.25	0.0660
2.25	0.0495
2.75	0.0372
3.00	0.0279
3.25	0.0208
3.50	0.0153
3.75	0.0111
4.00	0.0078
4.25	0.0052
4.50	0.0033
4.75	0.0018
5.00	0.0007

Figure 5.13: Results of calibrating the LIBOR model to the cap curve.

Maturities	$P(t_k, t_{k+1})$	Bond Volatilities
0	0.9977	Inf
0.25	0.9977	0.3980
0.50	0.9976	0.2896
0.75	0.9977	0.2437
1.00	0.9977	0.2176
1.25	0.9978	0.2009
1.50	0.9980	0.1897
1.75	0.9981	0.1821
2.00	0.9983	0.1771
2.25	0.9985	0.1743
2.25	0.9987	0.1735
2.75	0.9989	0.1746
3.00	0.9991	0.1778
3.25	0.9993	0.1838
3.50	0.9995	0.1932
3.75	0.9997	0.2080
4.00	0.9998	0.2317
4.25	0.9999	0.2746
4.50	1.0000	0.3767
4.75	1.0001	Inf
5.00		

Figure 5.14: Results of calibrating the Ho-Lee model to the bond curve.

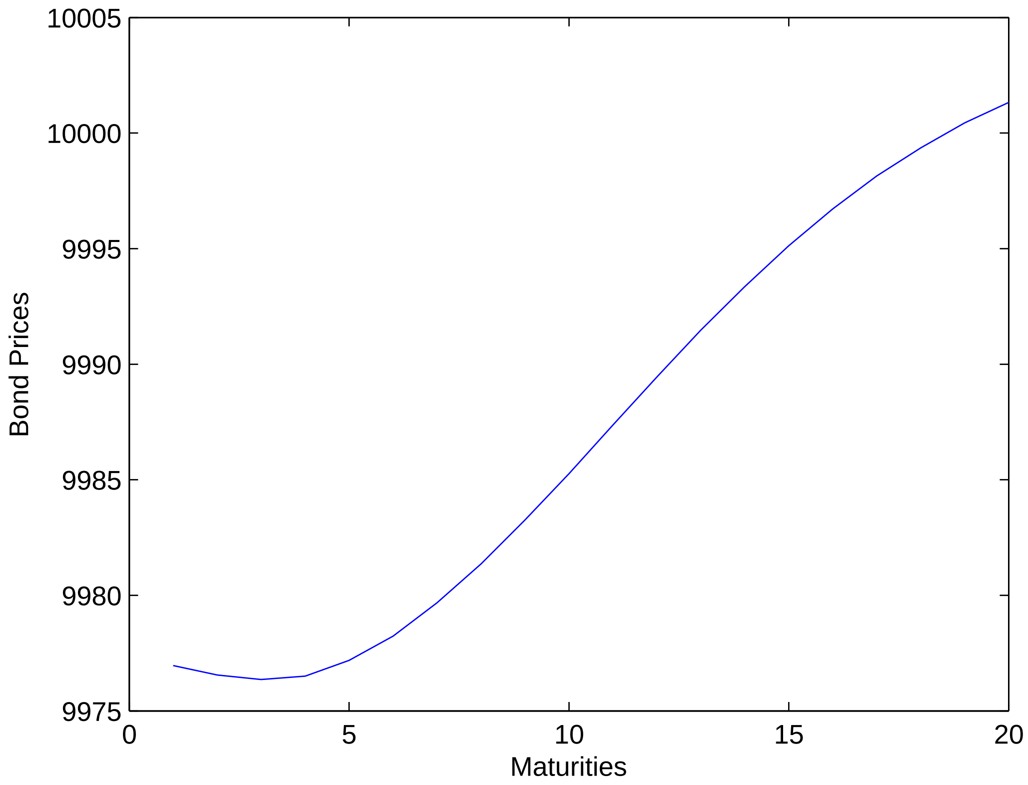

Figure 5.15: Time t bond prices as seen in column 2 of Fig. 5.13.

Payment Dates T_i	Forward Rates	Swaption values(non-LIBOR)	Swaption values(LIBOR)
1.5	0.0700	83.84	108.10
2	0.0750	148.59	191.60
2.5	0.0800	237.27	305.90
3	0.0850	348.88	449.80
3.5	0.0900	480.56	619.50
4	0.0950	628.50	810.20
4.5	0.1000	788.82	1016.90
5	0.1050	958.07	1235.10
5.5	0.1100	1133.47	1461.20
6	0.1150	1312.95	1692.60

Figure 5.16: Payer swaption values computed from the standard non-LIBOR market model and from the swap market model (LIBOR).

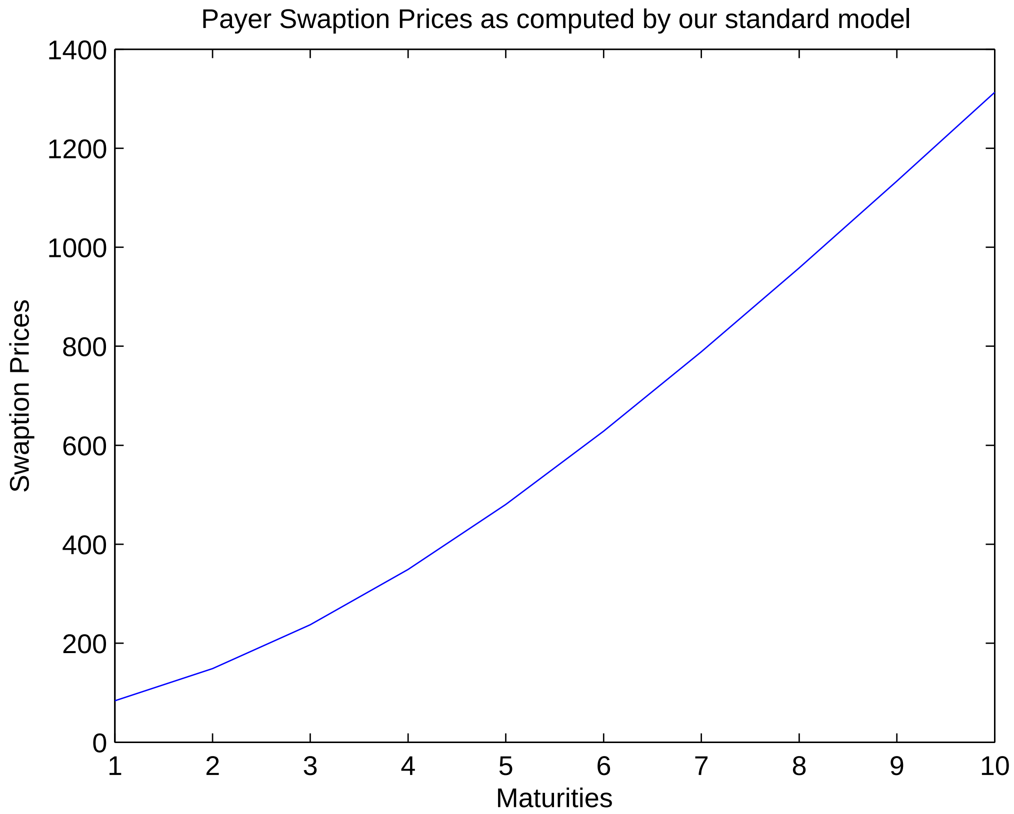

Figure 5.17: Payer swaption curve generated from the standard (non-LIBOR) market model. The curve results from plotting column 3 of Fig. 5.15 against maturities.

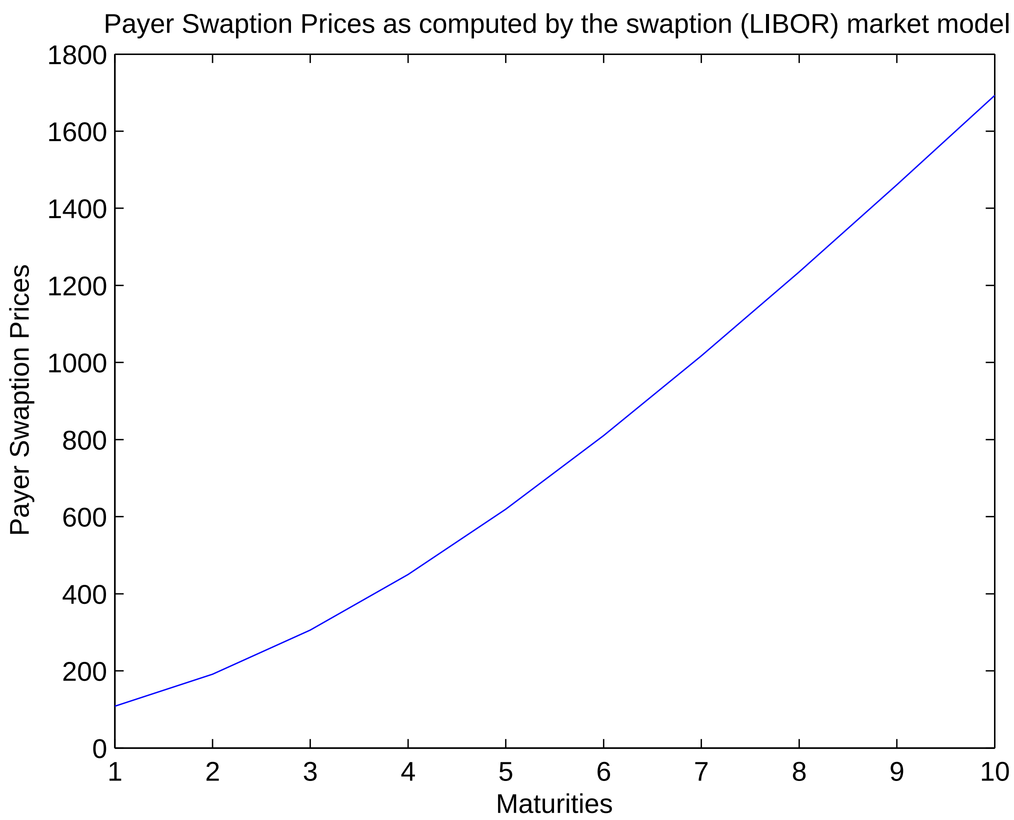

Figure 5.18: Payer swaption curve generated from the swaption (LIBOR) market model. The curve results from plotting column 4 of Fig 5.15 against maturities.

Chapter 6

New directions in interest rate theory

The slowness of Monte Carlo simulation within the LIBOR model framework is one of the main obstacles faced by market practitioners. To combat this, approximation formulae are widely used to price derivative instruments. In [33], Schellhorn and Chen suggest a new approach, the Double Layer Forward (DLF) simulation and show that the simulations in this scheme can be much faster than the traditionally used schemes. This methodology had earlier been applied to the swap market model by Jamshidian in [22]. An interesting observation is the fact that the approach by Schellhorn and Chen [33] can also most probably be extended to the valuing of risky bonds and credit sensitive derivatives as well as exchange rate derivatives.

Obviously, for coupon-bearing instruments, things change slightly. In [2], Baaquie proposed another new approach called the quantum field theory approach. Amongst its advantages over the Black's formula is the fact that a single volatility function can price a coupon bond option whereas in the Black's formula each caplet has its own volatility function. The quantum field approach looks very attractive and further research might be directed in extending this theory to other derivatives. This could include the risk management of both zero-coupon and coupon bonds. In actual fact, Baaquie and colleagues showed in [4] that using the quantum field theory, hedge parameters for risk management purposes of caps and floors can be provided.

In [3], Baaquie and Liang compare the Black's formula for a caplet/floorlet to the new field theory pricing formula, and they show that the field theory formulae have many advantages over Black's formulae. The market practice for pricing caplet/floorlet is the Black's formula. With the obvious advantages stated in [3], it remains to be seen how the market will react to this new approach. The quantum field theory approach seems appealing and further research could explore possibilities of its extension to the valuation of swaptions. Another direction of research could be a comparison of the field theory model to the LIBOR market model. This could be done by calibrating techniques.

Chapter 7

Conclusion

The present work has made a few notable contributions in the LIBOR market model. An explicit account of the theory underlying the forward risk-adjusted (neutral) valuation model was presented. This model is an improvement of the traditional securities risk-neutral valuation approach. Besides it being numerically labour intensive, it only needs a single data input, i.e the spot interest rate process. Future research could go a long way in trying to extend and apply this model in the pricing of a SAFEX-JIBAR cap/floor. Efforts should be directed in gaining confidence with the financial services companies so that they can release the much needed data for research purposes since they are the ultimate beneficiaries of such research results.

With the LIBOR market model now enjoying a pivotal role in modern interest rate derivative modeling, a detailed analysis of the relevant theory was presented. By analogue, this enabled an easy extension of the same ideas to the proposition of the JIBAR market model which, according to the numerical analytics, gives prices consistent with both economic practicality and with other models too. The biggest draw-back in this research was the unavailability of data. Data is a well-kept rare commodity among the competing players in the South African financial services sector. The unavailability of implied volatilities simply denies one the opportunity to follow the calibration route. See among others [9], [8] for calibration issues.

We showed that the swap and swaptions theory is much more complex than the cap and floor theory. Even though the two theories are generally incompatible, we showed that compatibility could be achieved if the swap life-span is partitioned into elementary quarterly periods. This reasoning enabled the proposition of the SAFEX-JIBAR swap and swaptions model on elementary periods.

Our numerical analytics suggest that a good quantitative analyst should not solely rely on one model. Besides not providing the trader with the best price, depending on one model exposes one to model risk also.

The results of calibrating the LIBOR model to the cap curve gave us the implied volatility structure appropriate to price the caplets, and hence the cap (Fig. 5.13). In the same manner, calibrating the Ho-Lee model to the bond curve (Fig 5.14). Figs. 5.17 and 5.18 show a similar shape for the payer swaption prices. However, the swaption values given by the non-LIBOR model are slightly lower than the LIBOR ones. This difference might be attributed to some minor input errors. Again, these results seem to support our earlier suggestions of combining models.

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